Statistical inference for Vasicek-type model driven by Hermite processes

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Content

1. Central and non-central limit theorems

2. Fractional Brownian motion

3. Hermite processes in a nutshell
   - Representation of Hermite processes
   - Properties of Hermite processes
   - Wiener integrals with respect to Hermite process

4. Vasicek-type model driven by Hermite processes
   - Motivation
   - Estimators for drift parameters
   - Fluctuations of the estimators
Central and non-central limit theorems

General framework:

- Let $X = (X_i)_{i \in \mathbb{N}}$ be a centered stationary Gaussian sequence with variance 1, and $\rho(i - j) = E[X_iX_j]$ be its covariance kernel.

- Let $g \in L^2(\mathbb{R}, \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx)$ such that $\int_{\mathbb{R}} g(x)e^{-\frac{x^2}{2}} dx = 0$, i.e.,

$$g(x) = \sum_{k=q}^{\infty} c_k H_k(x)$$

where $q \geq 1$, $a_q \neq 0$. Here $q$ is called the Hermite rank of $g$ and $H_k$ is the $k$-th Hermite polynomial

$H_1(x) = x$, $H_2(x) = x^2 - 1$, $H_3(x) = x^3 - 3x$, ....

Question:

$$\sum_{i=1}^{[Nt]} g(X_i) \longrightarrow ??? \quad \text{as } n \to \infty, \quad t \geq 0$$
Central and non-central limit theorems

1. Central limit theorem (Breuer-Major): When $X$ has short-range dependence (i.e., $\sum_{n \in \mathbb{Z}} |\rho(n)|^q < \infty$), then for all $q \geq 1$:

$$
\frac{1}{\sqrt{N}} \sum_{i=1}^{[Nt]} g(X_i) \xrightarrow{f.d.d.} \text{Brownian motion}
$$

2. Non-central limit theorem (Dobrushin-Major-Taqqu): When $X$ has long-range dependence (i.e., $\sum_{n \in \mathbb{Z}} |\rho(n)|^q = \infty$). Particular, $\rho(n) = E[X_0X_n] = n^{2H_0-2}L(n)$ for some $H_0 \in (1 - \frac{1}{2q}, 1)$ and $L$ a slowly varying function. Then,

$$
\frac{1}{N^H} \sum_{i=1}^{[Nt]} g(X_i) \xrightarrow{f.d.d.} \text{Hermite process } Z^{q,H}
$$

We have CLT for $q = 1$, limit behaviour $Z^{1,H}$ is the well-known fractional Brownian motion.
Fractional Brownian motion

**Definition**

Let $H \in (0, 1]$. A fractional Brownian motion (fBm in short) of Hurst parameter $H$ is a centered continuous Gaussian process $B^H = (B^H_t)_{t \geq 0}$ with covariance function

$$E[B^H_t B^H_s] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

- If $H = \frac{1}{2}$, that means $B^{\frac{1}{2}}$, is a classical Brownian motion.
- If $H = 1$, then $B^H_t = tB^H_1$ almost surely for all $t \geq 0$.

Since $H = 1$ is trivial, we always assume $H \in (0, 1)$.

Basic properties

Proposition

(i) **Self-similarity**: for all $a > 0, (a^{-H} B^H_{at})_{t \geq 0} \begin{equation} \text{law} \end{equation} (B^H_t)_{t \geq 0}$

(ii) **Stationary increments**: for all $h > 0, (B^H_{t+h} - B^H_h)_{t \geq 0} \begin{equation} \text{law} \end{equation} (B^H_t)_{t \geq 0}$

(iii) **Long-range dependence** $H > \frac{1}{2}$: $\sum_{n=1}^{\infty} E[B^H_1 (B^H_{n+1} - B^H_n)] = \infty$

(iv) **Hölder continuity**: the sample paths of $B^H$ are $\alpha$-Hölder continuous on each compact set for any $\alpha \in (0, H)$.

(v) fBm is neither a semimartingale nor a Markov process, except when its Hurst parameter is $\frac{1}{2}$ (that means, Brownian motion).

(vi) fBm has the form of a Volterra process, i.e., can be represented as $B^H_t = \int_0^t K_H(t, s) dW_s$, where $W = (W_t)_{t \geq 0}$ is classical Brownian motion and $K_H$ is an explicit square integrable kernel.
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(iii) **Long-range dependence** $H > \frac{1}{2}$: \[\sum_{n=1}^{\infty} E[B_1^H (B_{n+1}^H - B_n^H)] = \infty\]

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Figure 1. Paths of fBm for different values of $H$. 
Hermite processes in a nutshell


Where do Hermite processes come from?

Hermite processes appeared for the first time in a non-CLT proved by Taqqu (1975, 1979) and Dobrushin and Major (1979).

Recall, assume that:

- $g$ belongs to $L^2(\mathbb{R}, \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx)$ and satisfies $\int_{\mathbb{R}} g(x)e^{-\frac{x^2}{2}} dx = 0$ (that is, $g(x) = \sum_{k=q}^{\infty} c_k H_k(x)$ where $H_k$ denotes the $k$-th Hermite polynomial and $q$ is the Hermite rank of $g$).

- $X = (X_i)_{i \in \mathbb{Z}}$ is a centered stationary Gaussian seq. with variance 1, satisfying $E[X_0X_n] = n^{2H_0-2}L(n)$ for some $H_0 \in (1 - \frac{1}{2q}, 1)$ and $L$ a slowly varying function.

Then,

$$\frac{1}{N^H} \sum_{i=1}^{[Nt]} g(X_i) \xrightarrow{f.d.d.} "\text{Hermite process } Z_{t,}^{q,H}".$$  

Here $H = q(H_0 - 1) + 1$ belongs to $(\frac{1}{2}, 1)$ for all $q \geq 1$. 

Multiple Wiener-Itô integrals and Wiener chaos

Let \( B = B(h), h \in L^2(\mathbb{R}) \) be a Brownian field defined on a probability space \((\Omega, \mathcal{F}, P)\) satisfying \( E[B(h)B(g)] = \langle h, g \rangle_{L^2(\mathbb{R})}, \forall h, g \in L^2(\mathbb{R}).\)

For every \( q \geq 1 \), the \( q \)th Wiener chaos \( \mathcal{H}^B_q \) is defined as the closed linear subspace of \( L^2(\Omega) \) generated by the family of random variables

\[
\{H_q(B(h)), h \in L^2(\mathbb{R}), \|h\|_{L^2(\mathbb{R})} = 1\}
\]

The mapping \( I^B_q(h \otimes h) = H_q(B(h)) \) can be extended to a linear isometry between \( L^2_s(\mathbb{R}^q) \) and the \( q \)th Wiener chaos \( \mathcal{H}^B_q \).

When \( f \in L^2_s(\mathbb{R}^q) \), the random variable \( I^B_q(f) \) is called the multiple Wiener-Itô integral of \( f \) of order \( q \). One may write

\[
I^B_q(f) = \int_{\mathbb{R}^q} f(\xi_1, \ldots, \xi_p) dB_{\xi_1} \cdots dB_{\xi_p}
\]
Multiple Wiener-Itô integrals and Wiener chaos

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For every $q \geq 1$, the $q$th Wiener chaos $\mathcal{H}_q^B$ is defined as the closed linear subspace of $L^2(\Omega)$ generated by the family of random variables

$$\{ H_q(B(h)), h \in L^2(\mathbb{R}), \| h \|_{L^2(\mathbb{R})} = 1 \}$$

The mapping $I_q^B(h \otimes^q) = H_q(B(h))$ can be extended to a linear isometry between $L^2_s(\mathbb{R}^q)$ and the $q$th Wiener chaos $\mathcal{H}_q^B$. When $f \in L^2_s(\mathbb{R}^q)$, the random variable $I_q^B(f)$ is called the multiple Wiener-Itô integral of $f$ of order $q$. One may write

$$I_q^B(f) = \int_{\mathbb{R}^q} f(\xi_1, \ldots, \xi_p) dB_{\xi_1} \ldots dB_{\xi_p}$$
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When $f \in L^2_s(\mathbb{R}^q)$, the random variable $I^B_q(f)$ is called the multiple Wiener-Itô integral of $f$ of order $q$. One may write

$$I^B_q(f) = \int_{\mathbb{R}^q} f(\xi_1, \ldots, \xi_p) dB_{\xi_1} \ldots dB_{\xi_p}$$
For any $f \in L^2_s(\mathbb{R}^p)$ and $g \in L^2_s(\mathbb{R}^q)$, we have

- **Orthogonality - Isometry:**

$$E[I^B_p(f)I^B_q(g)] = \begin{cases} 
    p!\langle f, g \rangle_{L^2_s(\mathbb{R}^p)} & \text{if } p = q \\
    0 & \text{if } p \neq q.
\end{cases}$$

- **Hypercontractivity:** (All $L^k$-norms are equivalent)

$$E[|I^B_p(f)|^k]^{1/k} \leq (k - 1)^{p/2}E[|I^B_p(f)|^2]^{1/2} \text{ for any } k \in [2, \infty).$$

Hermite process viewed as a multiple Wiener-Itô integral

**Definition**

The Hermite process \((Z_{t}^{q,H})_{t \geq 0}\) of order \(q \geq 1\) and self-similarity parameter \(H \in (\frac{1}{2}, 1)\) is defined as

\[
Z_{t}^{q,H} = c(H, q) \int_{\mathbb{R}^q} \left( \int_{0}^{t} \prod_{j=1}^{q} (s - \xi_j)^{H_0 - \frac{3}{2}} ds \right) dB_{\xi_1} \ldots dB_{\xi_q},
\]

where \((B_t)_{t \in \mathbb{R}}\) is a two-sided standard Brownian motion and \(c(H, q)\) is calculated to ensure that \(E[(Z_{1}^{q,H})^2] = 1\).

\[
c(H, q) = \sqrt{\frac{H(2H - 1)}{q! \beta^q(H_0 - \frac{1}{2}, 2 - 2H_0)}}, \quad H_0 = 1 + \frac{H - 1}{q} \in \left(1 - \frac{1}{2q}, 1\right).
\]
Hermite random variable

Hermite process of order \( q = 1 \) is nothing but the fractional Brownian motion. fBm is the only Hermite process to be Gaussian (and that one could have defined for \( H \leq \frac{1}{2} \) as well).

**Definition**

A random variable which has the same law as \( Z_{1,q}^H \) is called a Hermite random variable of order \( q \) and parameter \( H \).

Hermite process of order \( q = 2 \) is called the Rosenblatt process. Denoted by \( R^H = (R^H_t)_{t \geq 0} \).

**Definition**

A random variable which has the same law as \( R^H_1 \) is called a Rosenblatt random variable of parameter \( H \).
Hermite random variable

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**Definition**

A random variable which has the same law as \( R_1^H \) is called a Rosenblatt random variable of parameter \( H \).
Properties of Hermite processes

The Hermite process $Z_{q,H}$ shares many properties with the fBm (corresponding to $q = 1$), EXCEPT GAUSSIANITY (for $q \geq 2$).

- **Self-similarity**
  
  For all $c > 0$, $(Z_{ct}^{q,H})_{t \geq 0} \overset{\text{law}}{=} (c^H Z_{t}^{q,H})_{t \geq 0}$.

- **Stationarity of increments**
  
  For any $h > 0$, $(Z_{t+h}^{q,H} - Z_{t}^{q,H})_{t \geq 0} \overset{\text{law}}{=} (Z_{t}^{q,H})_{t \geq 0}$.

- **Covariance function**
  
  For all $s, t \geq 0$, $E[Z_{t}^{q,H} Z_{s}^{q,H}] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})$.

- **Long-range dependence**
  
  $\sum_{n \geq 1} |E[Z_{1}^{q,H} (Z_{n+1}^{q,H} - Z_{n}^{q,H})]| = \infty$.

- **Hölder continuity**
  
  $Z_{q,H}$ admits a version with Hölder continuous sample paths of any order $\beta \in (0, H)$ on any compact interval.

- **Finite moments**
  
  For every $p \geq 1, t \geq 0$, $E[|Z_{t}^{q,H}|^p] \leq C_{p,q} t^{pH}$. 
Small digression: a mysterious conjecture on Rosenblatt distribution

Unfortunately, the explicit distribution of Rosenblatt random variable is not known (only numerical approximations are available).

Mysterious conjecture (Taqqu and Veillette 2013): whatever the value of $H$, it seems that

\[
P(R_1^H \leq -0.6256) = 0.2658, \quad P(R_1^H \leq 1.3552) = 0.9123.
\]
Wiener integrals with respect to Hermite process

- $\int_{\mathbb{R}} f(u) dZ_u^{q,H}$ is well-defined for any $f : \mathbb{R} \rightarrow \mathbb{R}$ such that
  $$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(u)f(v)||u - v|^{2H-2} dudv < \infty.$$

- Isometry $f \mapsto \int_{\mathbb{R}} f(u) dZ_u^{q,H}$:

  $$E \left[ \int_{\mathbb{R}} f(u) dZ_u^{q,H} \int_{\mathbb{R}} g(v) dZ_v^{q,H} \right] = H(2H - 1) \int_{\mathbb{R}} \int_{\mathbb{R}} f(u)g(v)|u - v|^{2H-2} dudv.$$
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Wiener integrals with respect to Hermite process

It can be expressed as a multiple Wiener-Itô integral

\[
\int_{\mathbb{R}} f(u) dZ_u^{q,H} = c(H, q) \int_{\mathbb{R}^q} \left( \int_{\mathbb{R}} f(u) \prod_{j=1}^{q} (u-\xi_j)_+^{H_0-\frac{3}{2}} du \right) dB_{\xi_1} \ldots dB_{\xi_q},
\]

with \( c(H, q) \) and \( H_0 \) given in (2).

Choosing \( f(u) = 1_{[0,t]}(u) \), then

\[
\int_{\mathbb{R}} 1_{[0,t]}(u) dZ_u^{q,H}
\]

\[
= c(H, q) \int_{\mathbb{R}^q} \left( \int_{\mathbb{R}} 1_{[0,t]}(u) \prod_{j=1}^{q} (u-\xi_j)_+^{H_0-\frac{3}{2}} du \right) dB_{\xi_1} \ldots dB_{\xi_q}
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\int_{\mathbb{R}} f(u) dZ_{u}^{q,H} = c(H, q) \int_{\mathbb{R}^{q}} \left( \int_{\mathbb{R}} f(u) \prod_{j=1}^{q} (u - \xi_{j})_{+}^{H_{0} - \frac{3}{2}} du \right) dB_{\xi_{1}} \ldots dB_{\xi_{q}},
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Vasicek-type model driven by Hermite processes - Motivation

- The (non-stationary) fractional Vasicek process is the unique strong solution of the Langevin equation driven by the fractional Brownian motion $B^H$

$$X_0 = 0, \quad dX_t = a(b - X_t)dt + \sigma dB^H_t, \quad t \geq 0,$$  \hspace{1cm} (3)

Here $a > 0$, $b \in \mathbb{R}$ are real drift parameters and $\sigma > 0$ is the volatility of the model.

- When $b = 0$, $X$ is a fractional Ornstein-Uhlenbeck process

- The fractional Vasicek model has received a lot of attention, because of its potential for modelling purpose and since one can use the powerful toolbox of Gaussian analysis to deal with it.
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$$X_0 = 0, \quad dX_t = a(b - X_t)dt + \sigma dB^H_t, \quad t \geq 0, \quad (3)$$

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The fractional Vasicek model has received a lot of attention, because of its potential for modelling purpose and since one can use the powerful toolbox of Gaussian analysis to deal with it.
Motivation

But in some situation, the Gaussian assumption may be implausible. For example in hydrology when we would like to analyze river-flow time series.

This is why we propose to look at a non-Gaussian extension:

\[ dX_t = a(b - X_t)dt + \sigma dZ_t^{q,H}, \quad X_0 = 0. \]

Here \( Z_t^{q,H} \) stands for a Hermite process.
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This is why we propose to look at a non-Gaussian extension:

\[ dX_t = a(b - X_t)dt + \sigma dZ_t^{q,H}, \quad X_0 = 0. \]

Here \( Z^{q,H} \) stands for a Hermite process.
The (non-stationary) Vasicek-type model driven by Hermite processes is the unique (pathwise) solution of the Langevin equation driven by Hermite process $Z^{q,H}$

$$dX_t = a(b - X_t)dt + \sigma dZ^{q,H}_t, \quad X_0 = 0.$$  \hspace{1cm} (4)

It is easily shown that the solution of (4) is given by

$$X_t = b(1 - e^{-at}) + \sigma \int_0^t e^{-a(t-u)} dZ^{q,H}_u.$$

The choice $q = 1$ in (4) corresponds to fractional Vasicek model. When $b = 0$, one gets Hermite Ornstein-Uhlenbeck process.
Vasicek-type model driven by Hermite processes

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Problem we have looked at:

Recall (4): \( dX_t = a(b - X_t)dt + \sigma dZ_t^{q,H} \).
Here \( H \in (\frac{1}{2}, 1) \), \( \sigma \) are known (assume \( \sigma = 1 \)), \( a \) and \( b \) are unknown.

- Construct estimators for \( a \) and \( b \).
- Study their asymptotic properties (their consistency as well as their fluctuations around the true value of the parameter) based on a continuous-time observation of \( X \).
- Do our estimators for \( a \) and \( b \) have the same asymptotic behavior when \( q = 1 \) (fBm case, fractional model (3)) and \( q \geq 2 \) (non-Gaussian case, model (4))? Answering this question is equivalent to understand whether the Gaussian feature of the fBm really matters when estimating the unknown parameters in the fractional Vasicek model (3).
Estimators for drift parameters

Recall (4): \[ dX_t = a(b - X_t)dt + dZ_{t}^{q,H}, \quad X_0 = 0. \]

Strong solution:

\[ X_t = b(1 - e^{-at}) + \int_{0}^{t} e^{-a(t-s)} dZ_{s}^{q,H}. \]

Assume: \( q \geq 1, H \in (\frac{1}{2}, 1) \) are known; \( a > 0 \) and \( b \in \mathbb{R} \) are unknown.

Definition

We define estimators for drift parameters \( a \) and \( b \) in (4) as follows:

\[ \hat{a}_T = \left( \frac{\alpha_T}{H \Gamma(2H)} \right)^{-\frac{1}{2H}}, \quad \text{where} \quad \alpha_T = \frac{1}{T} \int_{0}^{T} X_t^2 dt - \left( \frac{1}{T} \int_{0}^{T} X_t dt \right)^2, \]

\[ \hat{b}_T = \frac{1}{T} \int_{0}^{T} X_t dt. \]
Strong consistency of the estimators

**Theorem**

For any \( q \geq 1 \) and any \( H \in \left( \frac{1}{2}, 1 \right) \), we have, as \( T \to \infty \)

\[
\left( \hat{a}_T, \hat{b}_T \right) \overset{a.s.}{\to} (a, b).
\]

The proof of this theorem relies on the following proposition.

**Proposition**

For all \( q \geq 1 \) and \( H \in \left( \frac{1}{2}, 1 \right) \), one has, as \( T \to \infty \)

\[
\frac{1}{T} \int_0^T X_t dt \overset{a.s.}{\to} b
\]

\[
\frac{1}{T} \int_0^T X_t^2 dt \overset{a.s.}{\to} b^2 + a^{-2H} H \Gamma(2H).
\]
Strong consistency of the estimators

**Theorem**

For any $q \geq 1$ and any $H \in (\frac{1}{2}, 1)$, we have, as $T \to \infty$

$$(\hat{a}_T, \hat{b}_T) \xrightarrow{a.s.} (a, b).$$

The proof of this theorem relies on the following proposition.

**Proposition**

For all $q \geq 1$ and $H \in (\frac{1}{2}, 1)$, one has, as $T \to \infty$

$$\frac{1}{T} \int_0^T X_t dt \xrightarrow{a.s.} b$$ (6)

$$\frac{1}{T} \int_0^T X_t^2 dt \xrightarrow{a.s.} b^2 + a^{-2H} H \Gamma(2H).$$ (7)
Key steps of the proof of strong consistency

\[ X_t = h(t) + Y_t \] where \( h(t) = b(1 - e^{-at}) \), \( Y_t = \int_0^t e^{-a(t-u)} dZ_u^q,H \).

**Proof (6):** We write

\[
\frac{1}{T} \int_0^T X_t \, dt = \frac{b}{T} \int_0^T (1 - e^{-at}) \, dt + \frac{1}{T} \int_0^T Y_t \, dt.
\]

- Check that \( \frac{b}{T} \int_0^T (1 - e^{-at}) \, dt \rightarrow b \) (Lebesgue dominated convergence)
- Prove that \( \frac{1}{T} \int_0^T Y_t \, dt \rightarrow 0 \) almost surely
  
  Since \( \frac{1}{T} \int_0^T Y_t \, dt \) belongs to the qth Wiener chaos, it enjoys the hypercontractivity property + using Borel-Cantelli lemma to obtain desired conclusion.
Proof (7):

\[
\frac{1}{T} \int_0^T X_t^2 dt = \frac{1}{T} \int_0^T h(t)^2 dt + \frac{2}{T} \int_0^T h(t) Y_t dt + \frac{1}{T} \int_0^T Y_t^2 dt.
\]

1. Check \( \frac{1}{T} \int_0^T h(t)^2 dt \to b^2 \).

2. Prove \( \frac{1}{T} \int_0^T h(t) Y_t dt \overset{a.s.}{\to} 0 \). Hint: \( T^{-H} \int_0^T h(t) Y_t dt \overset{\text{law}}{\to} \frac{b}{a} Z_1^{q,H} \)

3. Show \( \frac{1}{T} \int_0^T Y_t^2 dt \overset{a.s.}{\to} a^{-2H} H \Gamma(2H) \).

Hint: First, observe that

\[
\frac{1}{T} \int_0^T E[Y_t^2] dt \to a^{-2H} H \Gamma(2H)
\]

Secondly, applying the next following results on quadratic functional of long memory moving average process \( (Y_t)_{t \geq 0} \).
Proof (7):

\[
\frac{1}{T} \int_0^T X_t^2 \, dt = \frac{1}{T} \int_0^T h(t)^2 \, dt + \frac{2}{T} \int_0^T h(t) Y_t \, dt + \frac{1}{T} \int_0^T Y_t^2 \, dt.
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   Hint: First, observe that

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   \]

   Secondly, applying the next following results on quadratic functional of long memory moving average process \((Y_t)_{t \geq 0}\)
Proof (7) Recall $Y_t = \int_0^t e^{-a(t-u)} dZ_u^q, H$. As $T \to \infty$, we have

- If $q \geq 2$ or ($q = 1$ and $H > \frac{3}{4}$):
  $$ T^\frac{2}{q}(1-H)-1 \int_0^T (Y_t^2 - E[Y_t^2]) dt \xrightarrow{\text{law}} \text{Rosenblatt r.v.} $$

- If $q = 1$ and $H \in (\frac{1}{2}, \frac{3}{4})$:
  $$ T^{-\frac{1}{2}} \int_0^T (Y_t^2 - E[Y_t^2]) dt \xrightarrow{\text{law}} \text{Standard Gaussian r.v.} $$

- If $q = 1$ and $H = \frac{3}{4}$
  $$ (T \log(T))^{-\frac{1}{2}} \int_0^T (Y_t^2 - E[Y_t^2]) dt \xrightarrow{\text{law}} \text{Standard Gaussian r.v.} $$

The asymptotic behaviour holds for general long memory moving average processes $Y_t = \int_0^t x(t - u) dZ_u^q, H$ satisfying that

$$ \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} |x(u)x(v)||u - v|^{2H-2} du dv < \infty. $$
Fluctuations of the estimators: non-Gaussian case

Theorem

Let $X = (X_t)_{t \geq 0}$ be the unique strong solution to

$$dX_t = a(b - X_t)dt + dZ_t^{q,H} \text{ with } a > 0, b \in \mathbb{R}.$$ 

For any $q \geq 2$ and any $H \in (\frac{1}{2}, 1)$:

$$\left(T_{\frac{2}{q}(1-H)}^{\frac{1}{q}} \{\hat{a}_T - a\}, T^{1-H} \{\hat{b}_T - b\}\right) \overset{\text{law}}{\to} \left(-\frac{a^{1-\frac{2}{q}(1-H)}}{2H^2\Gamma(2H)} G_\infty, \frac{1}{a}Z_{1,H}^{q,H}\right),$$

as $T \to \infty$. Here $G_\infty$ is distributed according to Rosenblatt distribution of parameter $1 + (2H - 2)/q$ up to an explicit constant. $Z_{1,H}^{q,H}$ is a Hermite random variable (Hermite process evaluated at 1).
Fluctuations of the estimators: Gaussian case

**Theorem**

- **Case** $q = 1$ and $H < \frac{3}{4}$:
  
  \[
  \left( \sqrt{T} \left\{ \hat{a}_T - a \right\}, T^{1-H} \left\{ \hat{b}_T - b \right\} \right) \xrightarrow{\text{law}} \left( -\frac{a^{1+4H}\sigma_H}{2H^2\Gamma(2H)} N, \frac{1}{a} N' \right).
  \]

- **Case** $q = 1$ and $H = \frac{3}{4}$:
  
  \[
  \left( \sqrt{\frac{T}{\log T}} \left\{ \hat{a}_T - a \right\}, T^{\frac{1}{4}} \left\{ \hat{b}_T - b \right\} \right) \xrightarrow{\text{law}} \left( \frac{3}{4} \sqrt{\frac{a}{\pi}} N, \frac{1}{a} N' \right).
  \]

- **Case** $q = 1$ and $H > \frac{3}{4}$:
  
  \[
  \left( T^{2-2H} \left\{ \hat{a}_T - a \right\}, T^{1-H} \left\{ \hat{b}_T - b \right\} \right) \xrightarrow{\text{law}} \left( -\frac{a^{2H-1}}{2H^2\Gamma(2H)} \left( G_\infty - (B_1^H)^2 \right), \frac{1}{a} B_1^H \right).
  \]
Proposition (Definition)

Assume either \((q = 1 \text{ and } H > \frac{3}{4})\) or \(q \geq 2\). Fix \(T > 0\). Let \(U_T = (U_T(t))_{t \geq 0}\) be a process defined as \(U_T(t) = \int_0^t e^{-T(t-u)}dZ_{u,H}^q\). Finally, let \(G_T\) be the random variable defined as

\[
G_T = T^{\frac{2}{q}(1-H)+2H} \int_0^1 (U_T(t)^2 - E[U_T(t)^2])dt.
\]

Then \(G_T\) converges in \(L^2(\Omega)\) to a limit written \(G_\infty\). Moreover, \(G_\infty/B_{H,q}\) is distributed according to the Rosenblatt distribution of parameter \(1 - \frac{2}{q}(1-H)\), where \(B_{H,q}\) is an explicit cst.
Keys of proof of fluctuations

Fluctuations of $\hat{b}_T$

\[
T^{1-H}\{\hat{b}_T - b\} = T^{1-H}\left\{ \frac{1}{T} \int_0^T Y_t dt - \frac{b}{T} \int_0^T e^{-at} dt \right\} = \frac{Z_{q,H}^T}{aT^H} + O(T^{-H})
\]

Fluctuations of $\hat{a}_T$

\[
T_{q}^{2(1-H)}\{\hat{a}_T - a\} = -\frac{a^{1+2H}}{2H^2(2H)} \left( T_{q}^{2(1-H)} A_T - T_{q}^{2(1-H)-2} \left( \frac{Z_{q,H}^T}{a^2} \right)^2 \right) + o(1),
\]

Where $A_T = \frac{1}{T} \int_0^T (Y_t^2 - E[Y_t^2]) dt$
Keys of proof of fluctuations

Fluctuations of \((\hat{a}_T, \hat{b}_T)\)

We consider first the case \((q = 1 \text{ and } H > \frac{3}{4})\) or \((q \geq 2)\). Since \(Z^{q,H}\) satisfies the scaling property, we can write

\[
\left( T^{\frac{2}{q}(1-H)} A_T, T^{-H} Z^{q,H}_T \right) \overset{\text{law}}{\rightarrow} \left( a^{-\frac{2}{q}(1-H)-2H} G_{aT}, Z^{q,H}_1 \right),
\]

and we deduce from the \(L^2\)-convergence of \(G_T\) that

\[
\left( T^{\frac{2}{q}(1-H)} A_T, T^{-H} Z^{q,H}_T \right) \overset{\text{law}}{\rightarrow} \left( a^{-\frac{2}{q}(1-H)-2H} G_\infty, Z^{q,H}_1 \right).
\]

For the Gaussian case \(q = 1\) and \(H \in (\frac{1}{2}, \frac{3}{4}]\): we use the powerful toolbox of Gaussian analysis to study fluctuation
Keys of proof of fluctuations

Fluctuations of \((\hat{a}_T, \hat{b}_T)\)

We consider first the case \((q = 1\) and \(H > \frac{3}{4}\)) or \((q \geq 2\)). Since \(Z^{q,H}\) satisfies the scaling property, we can write

\[
\left( T_{\frac{2}{q}(1-H)} A_T, T^{-H} Z_T^{q,H} \right) \overset{\text{law}}{=} \left( a^{-\frac{2}{q}(1-H)-2H} G_{\alpha_T}, Z_1^{q,H} \right),
\]

and we deduce from the \(L^2\)-convergence of \(G_T\) that

\[
\left( T_{\frac{2}{q}(1-H)} A_T, T^{-H} Z_T^{q,H} \right) \overset{\text{law}}{\rightarrow} \left( a^{-\frac{2}{q}(1-H)-2H} G_{\infty}, Z_1^{q,H} \right).
\]

For the Gaussian case \(q = 1\) and \(H \in \left( \frac{1}{2}, \frac{3}{4} \right]\): we use the powerful toolbox of Gaussian analysis to study fluctuation
Conclusion

Question: Do our estimators for $a$ and $b$ have the same asymptotic behavior in case of Gaussian and non-Gaussian ($q \geq 2$) or not?

Answer:

- The strong consistency of $(\hat{a}_T, \hat{b}_T)$ is universal for any Vasicek type model driven by Hermite process as a noise, no matter that it is Gaussian or not.

- The fluctuations of our estimators around the true value of the drift parameters depend heavily on the order $q$ and Hurst parameter $H$ of the underlying Hermite process.

This gives us some hints to understand how much the fractional Vasicek model relies on the Gaussian feature.
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