On the convex hull of random walks (and Lévy processes)

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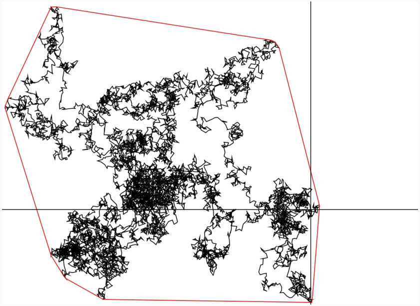
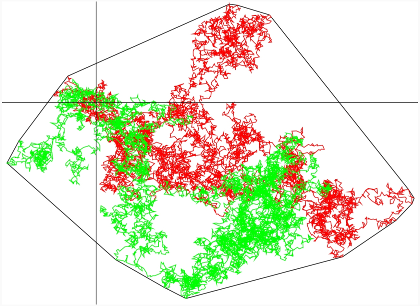
Motivations & Questions

Introduction: random walks, Brownian motion & animal movement





A convex hull-based estimator of home range sizes, Worton, *Biometrics* 51 (4) (1995)
Analyzing animal movements using Brownian bridges, Horne et al, *Ecology* 88 (9) (2007)
Home Range Estimates, Boyle *et al.*, *Folia Primatol.* 80 (2009)
Convex hull of N Brownian motions: Exact results & an application to ecology, R-F et al, PRL 103 (2009)
The Physics of Foraging, Viswanathan et al, Cambridge University Press (2011)
Modelling animal movement as Brownian bridges with covariates, Kranstauber, *Movement Ecology* 7 (1) (2019)

Introduction: motivations

Context(s), problems and motivations

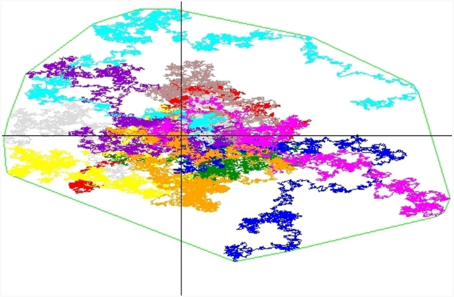
- Animal movement, dispersal and spatial ecology
- Transport in confined geometries, Polymer & protein conformation (biophysics)
- Extreme-value statistics (probability theory, physics, engineering, finance)
- Stochastic geometry (probability theory, imaging techniques)

Outline

- Perimeter & area via Cauchy formulae
- Number of edges/facets

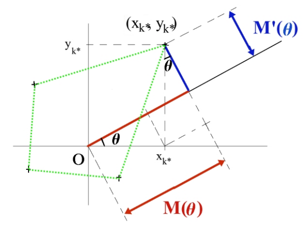
Perimeter & area using Cauchy formulae

Convex hull of n Brownian paths – Perimeter & area



- Only previuously known results were for $n = 1$
- Support function + Cauchy formulae $\Rightarrow$ general method for $n \geq 1$
  - Dimension $n \to$ Dimension $n - 1$
  - For 2D: maximum of a 1D walk and time at which it is attained
  - Exact results $\forall$ n, + asympotical behaviour for large n

Support function & Cauchy formulae



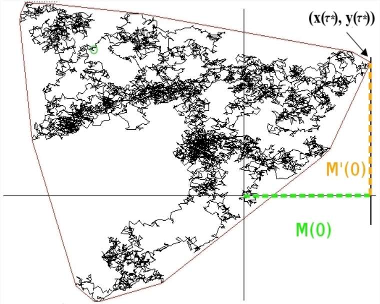
Perimeter of a closed convex curve

$$L = \int_0^{2\pi} d\theta \; M(\theta)$$

Area of a closed convex curve

$$A = \frac{1}{2} \int_0^{2\pi} d\theta \; \left[M^2(\theta) - \left[M'(\theta) \right]^2 \right]$$

Cauchy formulae for an (isotropic) Brownian path



$x(\tau)$, $y(\tau)$ **indep. 1D Brownians,** $0 \leq \tau \leq T$

Average perimeter

$$\langle L \rangle = 2\pi \langle M(0) \rangle$$

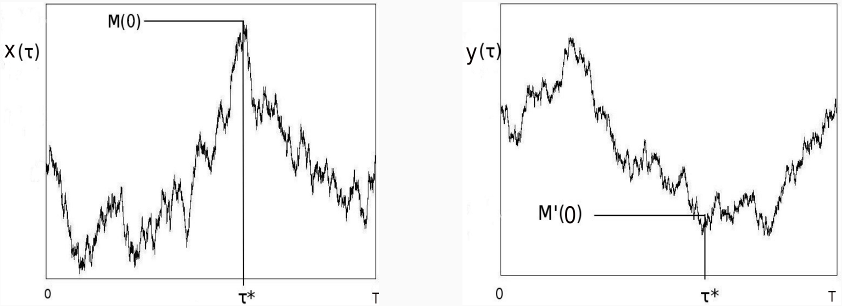
with $M(0) = \max_{0 \leq \tau \leq T} \{x(\tau)\} \equiv x(\tau^*)$

Average area

$$\langle A \rangle = \pi \left[\langle M^2(0) \rangle - \langle [M'(0)]^2 \rangle \right]$$

with $M'(0) = y(\tau^*)$

$M'(0)$ = value of y when x reaches its maximum

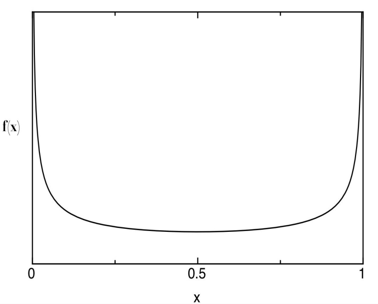


$$\langle M(0)\rangle = \int_0^\infty dM\ M\ \sigma(M)$$

$$\langle M^2(0)\rangle = \int_0^\infty dM\ M^2\ \sigma(M)$$

$$\langle [M'(0)]^2\rangle = \int_{-\infty}^\infty du \int_0^T d\tau^*\ \rho_1(\tau^*|T)\ u^2\ g\left(y(\tau^*) = u \mid y(0) = 0\right)$$

Distribution of the time of maximum for a free Brownian motion on $[0, T]$

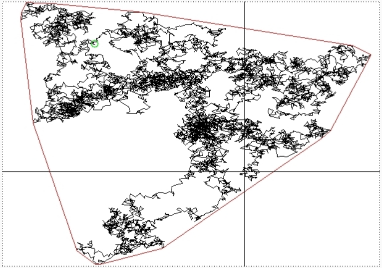


Arcsine law

$\rho_1(\tau^*|T) = \frac{1}{T}\ f\left(\frac{\tau^*}{T}\right)$

$$f(z) = \frac{1}{\pi\sqrt{z(1-z)}}$$

Results for $n = 1$ open Brownian path



$x(\tau)$, $y(\tau)$ **indep. 1D Brownian paths,** $0 \leq \tau \leq T$

Average perimeter

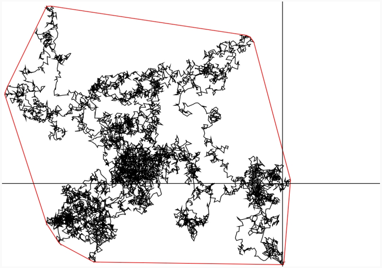
$$\langle L \rangle = \sqrt{8\pi T}$$

Average area

$$\langle A \rangle = \frac{\pi T}{2}$$

Takács, *Expected perimeter length*, Amer. Math. Month., 87 (1980)
El Bachir, *L'enveloppe convexe du mouvement brownien*, Thèse, Université Paul Sabatier, Toulouse (1983)

Results for $n = 1$ closed Brownian path



$x(\tau)$, $y(\tau)$ **indep. 1D Brownian bridges,** $0 \leq \tau \leq T$

Average perimeter

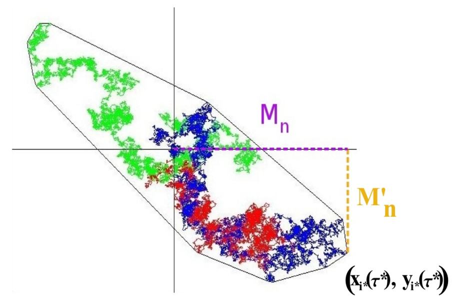
$$\langle L \rangle = \sqrt{\frac{\pi^3 T}{2}}$$

Average area

$$\boxed{\langle A \rangle = \frac{\pi T}{3}}$$

Goldman, *Le spectre de certaines mosaïques poissoniennes du plan et l'enveloppe convexe du pont brownien*, Prob. Theor. Relat. Fields, 105 (1996)

For n planar Brownian paths: maximum of n linear paths



$x_i(\tau)$, $y_i(\tau)$ indep. linear BMs, $0 \le \tau \le T$

Average perimeter

$$\langle L_n \rangle = 2\pi \langle M_n \rangle$$

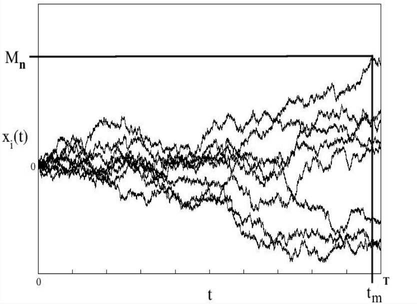
with $M_n = \max_{\tau,i} \{x_i(\tau)\} \equiv x_{i^*}(\tau^*)$

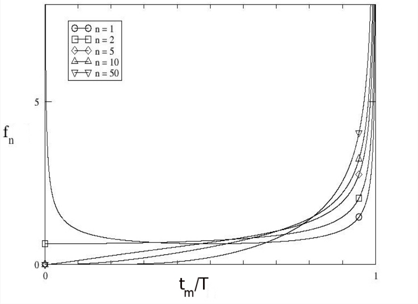
Average area

$$\langle A_n \rangle = \pi \left[\langle M_n^2 \rangle - \langle \left[M_n' \right]^2 \rangle \right]$$

with $M_n' = y_{i^*}(\tau^*)$

Global maximum for n indep. Brownian walkers



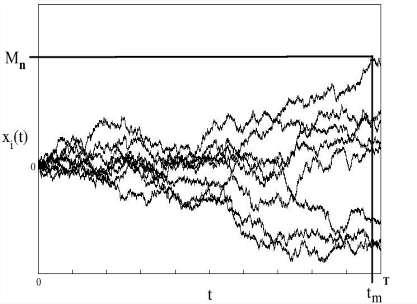


Cumulative distribution of global maximum M_n

$$\text{Prob}\left[M_n \leq M\right] \equiv F_n(M) = \left[\text{erf}\left(\frac{M}{\sqrt{2T}}\right)\right]^n$$

$\mathbf{erf}(M) = \frac{2}{\sqrt{\pi}} \int_0^M du\ e^{-u^2}$

Global maximum for n indep. Brownian walkers

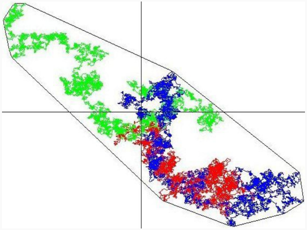


Distribution of the time t_m at which the global maximum is attained

$$\rho_n(t_m) = \frac{1}{T} f_n(z) = \frac{2n}{\pi T \sqrt{z(1-z)}} \int_0^\infty u\ e^{-u^2}\ [\mathrm{erf}(u\sqrt{z})]^{n-1}\ du$$

$z = \frac{t_m}{T}$

Convex hull of n indep. planar Brownian paths



$x_i(\tau)$, $y_i(\tau)$ **indep. 1D Brownian motions,**

$\mathbf{0} \leq \tau \leq T$

Average perimeter (open paths)

$\langle L_n \rangle = \alpha_n \sqrt{T}$

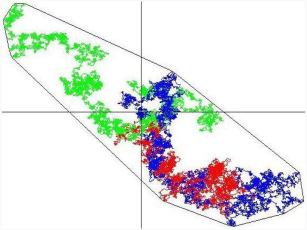
$$\alpha_n = 4n\sqrt{2\pi} \int_0^\infty du \; u \; e^{-u^2} \; [\mathrm{erf}(u)]^{n-1}$$

$$\alpha_1 = \sqrt{8\pi} = 5.013..$$

$$\alpha_2 = 4\sqrt{\pi} = 7.089..$$

$$\alpha_3 = 24 \frac{\tan^{-1}(1/\sqrt{2})}{\sqrt{\pi}} = 8.333..$$

Convex hull of n indep. planar Brownian paths



$x_i(\tau)$, $y_i(\tau)$ **indep. 1D Brownian motions,**

$0 \le \tau \le T$

Average area (open paths)

$\langle A_n \rangle = \beta_n T$

$$\beta_n = 4n\sqrt{\pi} \int_0^\infty du\ u\ [\mathrm{erf}(u)]^{n-1} \left(u e^{-u^2} - g(u) \right)$$

$$g(u) = \frac{1}{2\sqrt{\pi}} \int_0^1 \frac{e^{-u^2/t}\ dt}{\sqrt{t(1-t)}}$$

$$\begin{aligned} \beta_1 &= \frac{\pi}{2} = 1.570.. \\ \beta_2 &= \pi = 3.141.. \\ \beta_3 &= \pi + 3 - \sqrt{3} = 4.409.. \end{aligned}$$

Asymptotical behaviour

For n open paths:

$$\langle L_n \rangle \underset{n\to\infty}{\sim} 2\pi \sqrt{2T \ln n}$$

$$\langle A_n \rangle \underset{n\to\infty}{\sim} 2\pi\, T \ln n$$

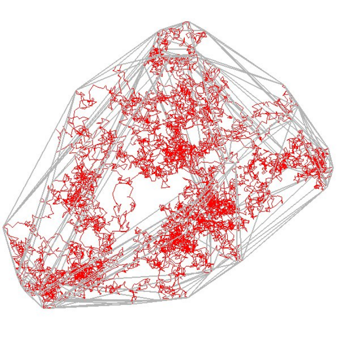
For n closed paths (bridges):

$$\langle L_n^c \rangle \underset{n\to\infty}{\sim} \pi \sqrt{2T \ln n}$$

$$\langle A_n^c \rangle \underset{n\to\infty}{\sim} \frac{\pi}{2}\, T \ln n$$

Number of edges/facets

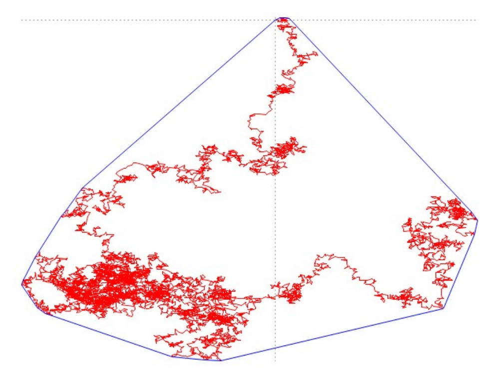
Edges & facets



Joint work with Dmitry Zaporozhets (St. Petersburg Dpt of the Steklov Institute)

Work supported by the Basis Foundation for Theoretical Physics and Mathematics

Convex hull of a single random walk

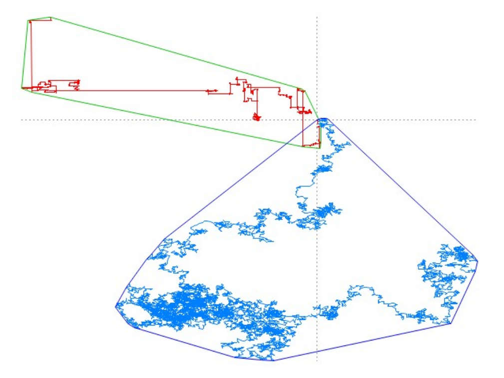


- In dimension 2, for a random walk with n steps:

$$\mathbb{E}\left[\left|\mathcal{F}\left(\mathcal{C}_2\right)\right|\right] = 2\sum_{k=1}^{n}\frac{1}{k}$$

Baxter (1961) *The Annals of Mathematical Statistics*, 32(3), 901-904.
Spitzer & Widom (1961) *Proceedings of the American Mathematical Society*, 12(3), 506-509.

Convex hull of a single random walk



- In dimension 2, for a random walk with n steps:

$$\mathbb{E}\left[\left|\mathcal{F}\left(\mathcal{C}_2\right)\right|\right] = 2\sum_{k=1}^{n}\frac{1}{k}$$

- whatever the (symmetric, continuous) distribution of the jumps

Baxter (1961) *The Annals of Mathematical Statistics*, 32(3), 901-904.
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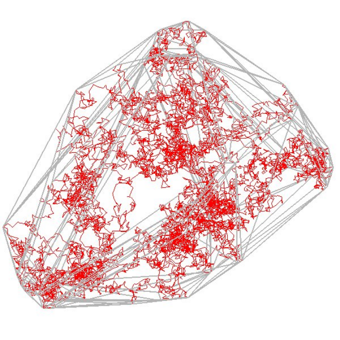
Convex hull of a single random walk



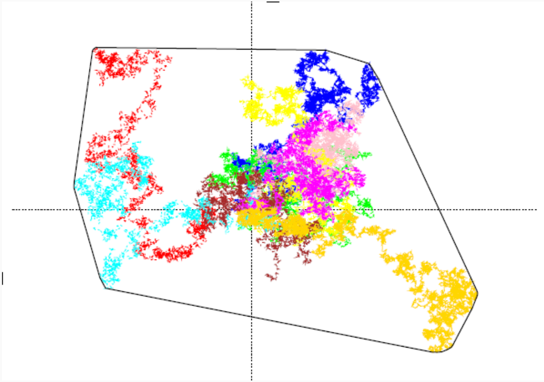
- In dimension d, for a random walk with n steps:

$$\mathbb{E}\left[\left|\mathcal{F}\left(\mathcal{C}_d\right)\right|\right] = 2 \sum_{\substack{j_1+\cdots+j_{d-1}\leq n \\ j_1,\ldots,j_{d-1}\geq 1}} \frac{1}{j_1 \cdot j_2 \cdot \ \ldots \ \cdot j_{d-1}},$$

- whatever the (symmetric, continuous) distribution of the jumps

Barndorff-Nielsen & Baxter (1963) *Transactions of the American Mathematical Society*, 108(2), 313-325.
Vysotsky & Zaporozhets (2018) *Transactions of the American Mathematical Society*, 370(11), 7985-8012.
Kabluchko, Vysotsky & Zaporozhets (2017) *Advances in Mathematics*, 320, 595-629.
R-F & Wespi (2017) *Physical Review E*, 95(3), 032129.

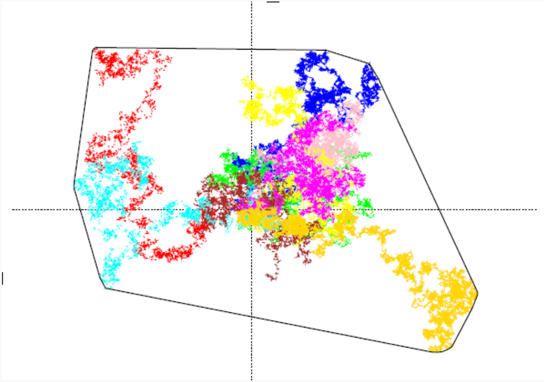
Convex hull of m random walks



- What about the global convex hull of multiple (independent) random walks?

R-F, Majumdar, & Comtet (2009) *Physical Review Letters*, 103(14), 140602.
R-F (2012) *Journal of Physics A: Mathematical and Theoretical*, 46(1), 015004.
Dewenter, Claussen, Hartmann, & Majumdar (2016) *Physical Review E*, 94(5), 052120.

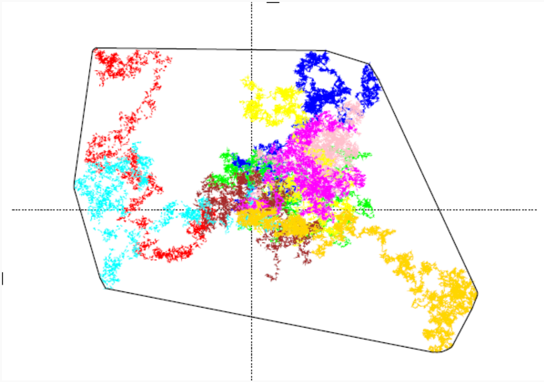
Convex hull of m random walks



- Expected number of edges on the boundary of the global convex hull?

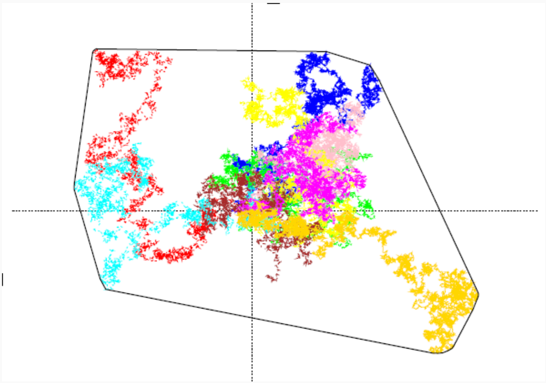
R-F (2012) *Journal of Physics A: Mathematical and Theoretical*, 46(1), 015004.

Convex hull of m random walks



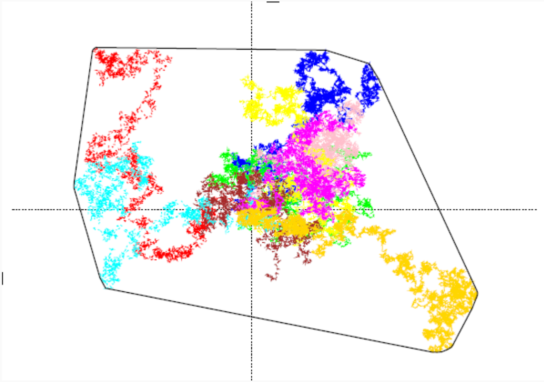
- Expected number of edges on the boundary of the global convex hull?
- More generally, in dimension d: expected number of faces?

Convex hull of m random walks



- Expected number of edges on the boundary of the global convex hull?
- More generally, in dimension d: expected number of faces?
- Not distribution-free: *eg* m single-step random walks $\longleftrightarrow$ m iid points (with 0)

Convex hull of m random walks



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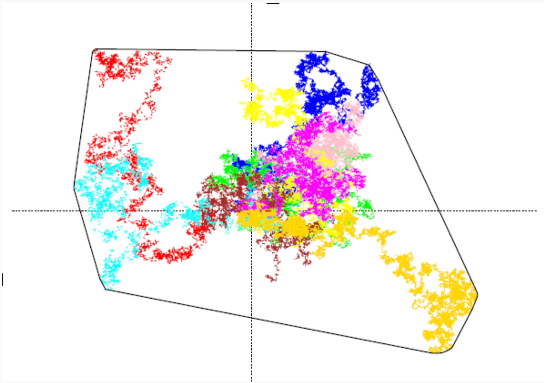
Efron (1965) *Biometrika*, 52(3/4), 331-343.
Rényi & Sulanke (1963, 1964) *Probability Theory and Related Fields*, 2(1), 75-84 & 3(2), 138-147.
Kabluchko & Zaporozhets (2018) *Transactions of the American Mathematical Society*, 372(3), 1709-1733.

Convex hull of m Gaussian random walks



- Expected number of edges on the boundary of the global convex hull?
- More generally, in dimension d: expected number of faces?
- Not distribution-free: *eg* m single-step random walks $\longleftrightarrow$ m iid points (with 0)

Efron (1965) *Biometrika*, 52(3/4), 331-343.
Rényi & Sulanke (1963, 1964) *Probability Theory and Related Fields*, 2(1), 75-84 & 3(2), 138-147.
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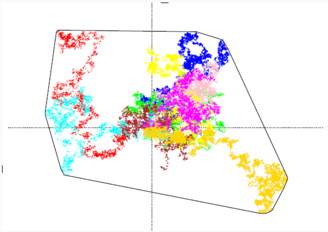
Setting

Setting

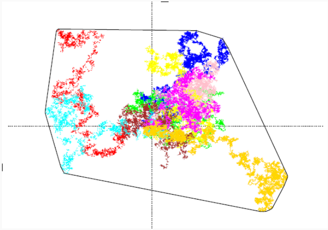


For $m, n_1, \dots, n_m \in \mathbb{N}$, let

$$X_1^{(1)}, \dots, X_{n_1}^{(1)}, \quad \dots, \quad X_1^{(m)}, \dots, X_{n_m}^{(m)}$$

be independent d-dimensional standard Gaussian vectors.

Setting



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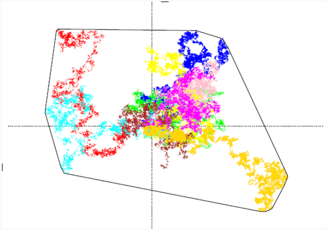
$$X_1^{(1)}, \dots, X_{n_1}^{(1)}, \quad \dots, \quad X_1^{(m)}, \dots, X_{n_m}^{(m)}$$

be independent d-dimensional standard Gaussian vectors.
Define the corresponding random walks

$$S_i^{(l)} = X_1^{(l)} + \dots + X_i^{(l)}, \quad 1 \le l \le m, \ 1 \le i \le n_l,$$

Setting

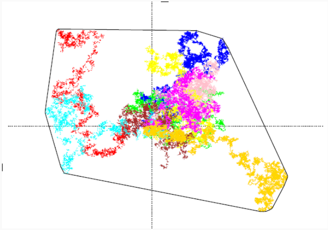


Define the corresponding random walks

$$S_i^{(l)} = X_1^{(l)} + \cdots + X_i^{(l)}, \quad 1 \le l \le m,\ 1 \le i \le n_l,$$

and a degenerate random walk $\left(S_i^{(0)}\right)_{i=1}^{1}$, with $S_1^{(0)} \equiv 0$.

Setting



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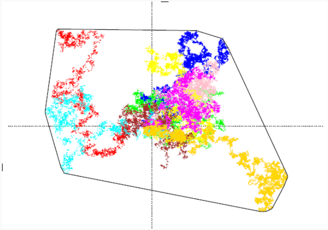
and a degenerate random walk $\left(S_i^{(0)}\right)_{i=1}^{1}$, with $S_1^{(0)} \equiv 0$.

The global convex hull is

$$\mathcal{C}_d = \operatorname{conv}\left\{S_1^{(0)}, S_1^{(1)}, \ldots, S_{n_1}^{(1)}, \ \ldots, \ \ S_1^{(m)}, \ldots, S_{n_m}^{(m)}\right\}.$$

Setting



- With probability one, $\mathcal{C}_d$ is a convex polytope with boundary of the form

$$\partial \mathcal{C}_d = \bigcup_{F \in \mathcal{F}(\mathcal{C}_d)} F,$$

where $\mathcal{F}(\mathcal{C}_d)$ stands for the set of $(d-1)$-dimensional faces of $\mathcal{C}_d$.

- Each face is a (d-1)-dimensional simplex almost surely.

Setting

- Let $k_0, \dots, k_m$ be integers s.t. $k_0 + \cdots + k_m = d$ and let $i_1^{(l)} < \cdots < i_{k_l}^{(l)} \leq n_l$ be indices, for those $l \in \{0, \dots, m\}$ s.t. $k_l > 0$.

Setting

- Let $k_0, \dots, k_m$ be integers s.t. $k_0 + \dots + k_m = d$ and let $i_1^{(l)} < \dots < i_{k_l}^{(l)} \leq n_l$ be indices, for those $l \in \{0, \dots, m\}$ s.t. $k_l > 0$.
- write S_d for the d-tuple

$$\mathsf{S}_d := \left(S_{k_0}^{(0)}, S_{i_1^{(1)}}^{(1)}, \dots, S_{i_{k_1}^{(1)}}^{(1)}, \dots, S_{i_1^{(m)}}^{(m)}, \dots, S_{i_{k_m}^{(m)}}^{(m)} \right)$$

with the convention that $\left\{ S_{i_1}^{(l)}, \dots, S_{i_{k_l}}^{(l)} \right\} := \emptyset$ whenever $k_l = 0$.

Setting

- Let $k_0, \ldots, k_m$ be integers s.t. $k_0 + \cdots + k_m = d$ and let $i_1^{(l)} < \cdots < i_{k_l}^{(l)} \leq n_l$ be indices, for those $l \in \{0, \ldots, m\}$ s.t. $k_l > 0$.
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- also write

$$\operatorname{conv} \mathrm{S}_d := \operatorname{conv} \left\{ S_{k_0}^{(0)}, S_{i_1^{(1)}}^{(1)}, \ldots, S_{i_{k_1}^{(1)}}^{(1)}, \ldots, S_{i_1^{(m)}}^{(m)}, \ldots, S_{i_{k_m}^{(m)}}^{(m)} \right\}.$$

Setting

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- also write

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Note that

- $\operatorname{conv} \mathsf{S}_d$ may be or not be a face of $\mathcal{C}_d$,
- every face can be represented as some $\operatorname{conv} \mathsf{S}_d$.

Setting

- write S_d for the d-tuple

$$\mathsf{S}_d := \left(S^{(0)}_{k_0}, S^{(1)}_{i^{(1)}_1}, \ldots, S^{(1)}_{i^{(1)}_{k_1}}, \ldots, S^{(m)}_{i^{(m)}_1}, \ldots, S^{(m)}_{i^{(m)}_{k_m}} \right)$$

- $\operatorname{conv} \mathsf{S}_d$ may be or not be a face of $\mathcal{C}_d$,
- every face can be represented as some $\operatorname{conv} \mathsf{S}_d$.
- Hence the crucial, albeit elementary, relation:

$$\sum_{F \in \mathcal{F}(\mathcal{C}_d)} g(F) = \sum_{\substack{k_0 + \cdots + k_m = d \\ 0 \le k_l \le n_l,\, l = 0, \ldots, m}} \sum_{\substack{1 \le i^{(l)}_1 < \cdots < i^{(l)}_{k_l} \le n_l \\ l = 0, \ldots, m \,:\, k_l > 0}} g(\mathsf{S}_d) \mathbb{I}_{\{\mathsf{S}_d \in \mathcal{F}(\mathcal{C}_d)\}} \text{ a.s.,}$$

with $g : \mathbb{R}^d \to \mathbb{R}^1$ an arbitrary, symmetric, non-negative, measurable function.

Setting

- write S_d for the d-tuple

$$\mathsf{S}_d := \left(S^{(0)}_{k_0}, S^{(1)}_{i^{(1)}_1}, \ldots, S^{(1)}_{i^{(1)}_{k_1}}, \ldots, S^{(m)}_{i^{(m)}_1}, \ldots, S^{(m)}_{i^{(m)}_{k_m}} \right)$$

- $\operatorname{conv} \mathsf{S}_d$ may be or not be a face of $\mathcal{C}_d$,
- every face can be represented as some $\operatorname{conv} \mathsf{S}_d$.
- Hence the crucial, albeit elementary, relation:

$$\sum_{F \in \mathcal{F}(\mathcal{C}_d)} g(F) = \sum_{\substack{k_0+\cdots+k_m=d \\ 0 \le k_l \le n_l,\, l=0,\ldots,m}} \sum_{\substack{1 \le i^{(l)}_1 < \cdots < i^{(l)}_{k_l} \le n_l \\ l=0,\ldots,m \,:\, k_l > 0}} g(\mathsf{S}_d) \mathbb{I}_{\{\mathsf{S}_d \in \mathcal{F}(\mathcal{C}_d)\}} \text{ a.s.},$$

  with $g : \mathbb{R}^d \to \mathbb{R}^1$ an arbitrary, symmetric, non-negative, measurable function.
- relevant choices of g will yield our results

Setting – some more notations

- Unconditional and conditional Gaussian persistence probabilities:
  $p_n(r) := \mathbb{P}\big[\sum_{i=1}^{k} N_i \le r,\ k = 1, \ldots, n\big]$,
  $q_n(r) := \mathbb{P}\big[\sum_{i=1}^{k} N_i \le r, k = 1, \ldots, n \,\big|\, \sum_{i=1}^{n} N_i = r\big]$,
  where $N_1, \ldots, N_n \in \mathbb{R}^1$ are independent standard Gaussian random variables.
  By symmetry of the distribution:
  $q_n(r) := \mathbb{P}\big[\sum_{i=1}^{k} N_i \ge 0, k = 1, \ldots, n \,\big|\, \sum_{i=1}^{n} N_i = r\big]$.

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  Note that

$$p_1(r) = \Phi(r) \quad \text{and} \quad q_1(r) = 1 \ \forall r \geq 0,$$

$$p_n(0) = \frac{(2n-1)!!}{(2n)!!} \quad \text{and} \quad q_n(0) = \frac{1}{n},$$

  where $\Phi(r)$ is the cdf of the standard Gaussian law, and the 3rd & 4th points were established by Sparre Andersen.

Setting – some more notations

- Unconditional and conditional Gaussian persistence probabilities:
  $p_n(r) := \mathbb{P}\big[\sum_{i=1}^{k} N_i \le r,\ k = 1, \dots, n\big]$,
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  By symmetry of the distribution:
  $q_n(r) := \mathbb{P}\big[\sum_{i=1}^{k} N_i \ge 0, k = 1, \dots, n \,|\, \sum_{i=1}^{n} N_i = r\big]$.
- P_d is the orthogonal projection onto the first $d - 1$ coordinates.
- $|\cdot|$ denotes volume or cardinality. $\kappa_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$ is the volume of the d-dimensional unit ball.
- $\mathcal{Q}$ is a matrix chosen uniformly from the orthogonal group $O(d)$, independently with the random walks.

Results (1)

A general formula

Theorem

For $g : \mathbb{R}^d \to \mathbb{R}^1$ a bounded measurable function, symmetric and invariant with respect to translations,

$$
\begin{aligned}
&\mathbb{E}\left[g(\mathsf{S}_d)\mathbb{I}_{\{\mathrm{conv}\,\mathsf{S}_d \in \mathcal{F}(\mathcal{C}_d)\}}\right] = \\
&\quad d!\,\kappa_d\,(2\pi)^{-d/2} \times \mathbb{E}[g(\mathcal{Q}\mathsf{T}_{d-1}) \cdot |\mathrm{conv}\,\mathsf{T}_{d-1}|] \\
&\quad \times \prod_{l\,:\,k_l \neq 0} \left[\frac{(2(n_l - i_{k_l}^{(l)}) - 1)!!}{(2(n_l - i_{k_l}^{(l)}))!!} \left(i_1^{(l)} \left(i_2^{(l)} - i_1^{(l)}\right)^3 \cdots \left(i_{k_l}^{(l)} - i_{k_l-1}^{(l)}\right)^3 \right)^{-1/2} \right] \\
&\quad \times \Bigg\{ \mathbb{I}_{\{k_0=0\}} \int_0^\infty \Bigg[\prod_{\substack{l\,:\,k_l=0 \\ l \neq 0}} p_{n_l}(r) \Bigg] \Bigg[\prod_{l\,:\,k_l \neq 0} q_{i_1^{(l)}}(r) \Bigg] \exp\Bigg(-\frac{r^2}{2} \sum_{l\,:\,k_l \neq 0} \frac{1}{i_1^{(l)}} \Bigg) \mathrm{d}r \\
&\qquad + \sqrt{2\pi}\,\mathbb{I}_{\{k_0=1\}} \prod_{l\,:\,k_l=0} \frac{(2n_l - 1)!!}{(2n_l)!!} \prod_{\substack{l\,:\,k_l \neq 0 \\ l \neq 0}} \frac{1}{i_1^{(l)}} \Bigg\}.
\end{aligned}
$$

where $\mathsf{T}_{d-1} \sim P_d \mathsf{S}_d$ is a (d-1)-simplex defined from the same indices as S_d.

Face probability

- Applying the previous theorem to $g \equiv 1$ leads to:

Theorem

$$\mathbb{P}\left[\operatorname{conv} \mathsf{S}_d \in \mathcal{F}(\mathcal{C}_d)\right] = d!\,\kappa_d\,(2\pi)^{-d/2} \times \mathbb{E}|\operatorname{conv} \mathsf{T}_{d-1}|$$

$$\times \prod_{l\,:\,k_l \neq 0} \left[\frac{(2(n_l - i^{(l)}_{k_l}) - 1)!!}{(2(n_l - i^{(l)}_{k_l}))!!} \left(i^{(l)}_1 \left(i^{(l)}_2 - i^{(l)}_1\right)^3 \dots \left(i^{(l)}_{k_l} - i^{(l)}_{k_l - 1}\right)^3 \right)^{-1/2} \right]$$

$$\times \Bigg\{ \mathbb{I}_{\{k_0 = 0\}} \int_0^\infty \left[\prod_{\substack{l\,:\,k_l = 0 \\ l \neq 0}} p_{n_l}(r) \right] \left[\prod_{l\,:\,k_l \neq 0} q_{i^{(l)}_1}(r) \right] \exp\left(-\frac{r^2}{2} \sum_{l\,:\,k_l \neq 0} \frac{1}{i^{(l)}_1} \right) \mathrm{d}r$$

$$+ \sqrt{2\pi}\, \mathbb{I}_{\{k_0 = 1\}} \prod_{l\,:\,k_l = 0} \frac{(2n_l - 1)!!}{(2n_l)!!} \prod_{\substack{l\,:\,k_l \neq 0 \\ l \neq 0}} \frac{1}{i^{(l)}_1} \Bigg\}.$$

Expected number of faces

- Summing the previous formula over all choices of k_l's and i_j's leads to:

Theorem

$$\mathbb{E}\,|\mathcal{F}(\mathcal{C}_d)| \;=\; d!\,\kappa_d\,(2\pi)^{-d/2} \sum_{\substack{k_0,\dots,k_m\geq 0\\ k_0\leq n_0,\dots,k_m\leq n_m\\ k_0+\cdots+k_m=d}} \; \sum_{\substack{1\leq i_1^{(l)}<\cdots<i_{k_l}^{(l)}\leq n_l\\ l=0,\dots,m\,:\,k_l>0}} \mathbb{E}|\mathrm{conv}\,\mathsf{T}_{d-1}|$$

$$\times \prod_{l\,:\,k_l\neq 0}\left[\frac{(2(n_l-i_{k_l}^{(l)})-1)!!}{(2(n_l-i_{k_l}^{(l)}))!!}\left(i_1^{(l)}\left(i_2^{(l)}-i_1^{(l)}\right)^3\cdots\left(i_{k_l}^{(l)}-i_{k_l-1}^{(l)}\right)^3\right)^{-1/2}\right]$$

$$\times\left\{\mathbb{I}_{\{k_0=0\}}\int_0^\infty\left[\prod_{\substack{l\,:\,k_l=0\\ l\neq 0}} p_{n_l}(r)\right]\left[\prod_{l\,:\,k_l\neq 0} q_{i_1^{(l)}}(r)\right]\exp\left(-\frac{r^2}{2}\sum_{l\,:\,k_l\neq 0}\frac{1}{i_1^{(l)}}\right)\mathrm{d}r\right.$$

$$\left.+\sqrt{2\pi}\,\mathbb{I}_{\{k_0=1\}}\prod_{l\,:\,k_l=0}\frac{(2n_l-1)!!}{(2n_l)!!}\prod_{\substack{l\,:\,k_l\neq 0\\ l\neq 0}}\frac{1}{i_1^{(l)}}\right\}.$$

Expected surface area of the boundary

- Applying the main theorem to $g(\mathsf{S}_d) = |\mathrm{conv}\,\mathsf{S}_d|$, we obtain the expected surface area (i.e. $(d-1)$-dimensional content) of the boundary of the convex hull, $\partial\mathcal{C}_d$:

Theorem

$$\mathbb{E}\,|\partial\mathcal{C}_d| \;=\; d!\,\kappa_d\,(2\pi)^{-d/2} \sum_{\substack{k_0,\dots,k_m\geq 0\\ k_0\leq n_0,\dots,k_m\leq n_m\\ k_0+\dots+k_m=d}} \;\sum_{\substack{1\leq i_1^{(l)}<\dots<i_{k_l}^{(l)}\leq n_l\\ l=0,\dots,m\,:\,k_l>0}} \mathbb{E}|\mathrm{conv}\,\mathsf{T}_{d-1}|^2$$
$$\times \prod_{l\,:\,k_l\neq 0}\left[\frac{(2(n_l-i_{k_l}^{(l)})-1)!!}{(2(n_l-i_{k_l}^{(l)}))!!}\left(i_1^{(l)}(i_2^{(l)}-i_1^{(l)})^3\dots(i_{k_l}^{(l)}-i_{k_l-1}^{(l)})^3\right)^{-1/2}\right]$$
$$\times\Bigg\{\mathbb{I}_{\{k_0=0\}}\int_0^\infty\Bigg[\prod_{\substack{l\,:\,k_l=0\\ l\neq 0}} p_{n_l}(r)\Bigg]\Bigg[\prod_{l\,:\,k_l\neq 0} q_{i_1^{(l)}}(r)\Bigg]\exp\Bigg(-\frac{r^2}{2}\sum_{l\,:\,k_l\neq 0}\frac{1}{i_1^{(l)}}\Bigg)\mathrm{d}r$$
$$+\sqrt{2\pi}\,\mathbb{I}_{\{k_0=1\}}\prod_{l\,:\,k_l=0}\frac{(2n_l-1)!!}{(2n_l)!!}\prod_{\substack{l\,:\,k_l\neq 0\\ l\neq 0}}\frac{1}{i_1^{(l)}}\Bigg\}.$$

Expected d-dimensional volume of the convex hull

- Recalling the Cauchy surface area formula: $\mathbb{E}|\mathcal{C}_{d-1}| = \frac{\kappa_{d-1}}{d\kappa_d}\mathbb{E}|\partial\mathcal{C}_d|$ leads to a formula for the expected (d-dimensional) volume of $\mathcal{C}_d$

Theorem

$$\mathbb{E}\,|\mathcal{C}_d| = d!\,\kappa_d\,(2\pi)^{-(d+1)/2} \sum_{\substack{k_0,\dots,k_m\ge 0\\ k_0\le n_0,\dots,k_m\le n_m\\ k_0+\dots+k_m=d+1}} \sum_{\substack{1\le i_1^{(l)}<\dots<i_{k_l}^{(l)}\le n_l\\ l=0,\dots,m\,:\,k_l>0}} \mathbb{E}|\mathrm{conv}\,\mathsf{T}_d|^2$$
$$\times \prod_{l\,:\,k_l\neq 0}\left[\frac{(2(n_l-i_{k_l}^{(l)})-1)!!}{(2(n_l-i_{k_l}^{(l)}))!!}\left(i_1^{(l)}(i_2^{(l)}-i_1^{(l)})^3\dots(i_{k_l}^{(l)}-i_{k_l-1}^{(l)})^3\right)^{-1/2}\right]$$
$$\times\Bigg\{\mathbb{I}_{\{k_0=0\}}\int_0^\infty\Bigg[\prod_{\substack{l\,:\,k_l=0\\ l\neq 0}} p_{n_l}(r)\Bigg]\Bigg[\prod_{l\,:\,k_l\neq 0} q_{i_1^{(l)}}(r)\Bigg]\exp\Bigg(-\frac{r^2}{2}\sum_{l\,:\,k_l\neq 0}\frac{1}{i_1^{(l)}}\Bigg)\mathrm{d}r$$
$$+\sqrt{2\pi}\,\mathbb{I}_{\{k_0=1\}}\prod_{l\,:\,k_l=0}\frac{(2n_l-1)!!}{(2n_l)!!}\prod_{\substack{l\,:\,k_l\neq 0\\ l\neq 0}}\frac{1}{i_1^{(l)}}\Bigg\}.$$

Expected number of facets containing the origin

- With $\mathcal{F}^0(\cdot) =$ the set of facets containing the origin as a vertex,
- one obtains a **distribution-free formula**

Theorem

$$\mathbb{E}|\mathcal{F}^0(\mathcal{C}_d)| = 2 \sum_{\substack{k_1+\cdots+k_m=d-1 \\ 0\le k_l\le n_l,\ l=1,\ldots,m}} \sum_{\substack{1\le i_1^{(l)}<\cdots<i_{k_l}^{(l)}\le n_l \\ l=1,\ldots,m\,:\,k_l>0}} \prod_{l=1}^{m}\left[\frac{1}{i_1^{(l)}}\frac{1}{i_2^{(l)}-i_1^{(l)}}\cdots\frac{1}{i_{k_l}^{(l)}-i_{k_l-1}^{(l)}}\frac{(2(n_l-i_{k_l}^{(l)})-1)!!}{(2(n_l-i_{k_l}^{(l)}))!!}\right]$$

Proofs – main ingredient(s)

Affine Blaschke-Petkantschin formula

- Let $\mathbb{S}^{d-1}$ be the unit $(d-1)-$dimensional sphere, centered at the origin and equipped with the Lebesgue measure μ normalized to be probabilistic.
- For $u \in \mathbb{S}^{d-1}$, let $u^\perp$ be the linear hyperplane orthogonal to u.

Then, for any non-negative measurable function $h : (\mathbb{R}^d)^d \to \mathbb{R}$,

$$
\begin{aligned}
&\int_{(\mathbb{R}^d)^d} h(\mathrm{x}_1, \ldots, \mathrm{x}_d)\mathrm{d}\mathrm{x}_1 \ldots \mathrm{d}\mathrm{x}_d \\
&\qquad = d!\, \kappa_d \int_{\mathbb{S}^{d-1}} \int_0^\infty \int_{(\mathrm{u}^\perp + r\mathrm{u})^d} h(\mathrm{x}_1, \ldots, \mathrm{x}_d)\, |\mathrm{conv}\,(\mathrm{x}_1, \ldots, \mathrm{x}_d)| \\
&\qquad\qquad\qquad \times \lambda_{\mathrm{u}^\perp}(\mathrm{d}\mathrm{x}_1) \ldots \lambda_{\mathrm{u}^\perp}(\mathrm{d}\mathrm{x}_d)\, \mathrm{d}r\, \mathrm{d}\mu(\mathrm{d}\mathrm{u}) \\
&\qquad = d!\, \kappa_d \int_{\mathbb{S}^{d-1}} \int_0^\infty \int_{(\mathrm{u}^\perp)^d} h(\mathrm{x}_1 + r\mathrm{u}, \ldots, \mathrm{x}_d + r\mathrm{u})\, |\mathrm{conv}\,(\mathrm{x}_1, \ldots, \mathrm{x}_d)| \\
&\qquad\qquad\qquad \times \lambda_{\mathrm{u}^\perp}(\mathrm{d}\mathrm{x}_1) \ldots \lambda_{\mathrm{u}^\perp}(\mathrm{d}\mathrm{x}_d)\, \mathrm{d}r\, \mathrm{d}\mu(\mathrm{d}\mathrm{u}).
\end{aligned}
$$

see *eg* Schneider & Weil (2008). *Stochastic and integral geometry*, Springer, Berlin.

Affine Blaschke-Petkantschin formula

We apply the B-P formula to compute:

$$\mathbb{E}[g(\mathsf{S}_d)\mathbb{I}_{\{\mathrm{conv}\,\mathsf{S}_d\in\mathcal{F}(\mathcal{C}_d)\}}] = \int_{(\mathbb{R}^d)^d} \mathbb{P}[\mathrm{conv}\,\mathsf{S}_d \in \mathcal{F}(\mathcal{C}_d)\,|\,\mathsf{S}_d = (\mathsf{x}_1, \dots, \mathsf{x}_d)]$$
$$\times\, g(\mathsf{x}_1, \dots, \mathsf{x}_d) f_{\mathsf{S}_d}(\mathsf{x}_1, \dots, \mathsf{x}_d)\mathrm{d}\mathsf{x}_1 \dots \mathrm{d}\mathsf{x}_d,$$

where f_{S_d} is the joint density of S_d.

Affine Blaschke-Petkantschin formula

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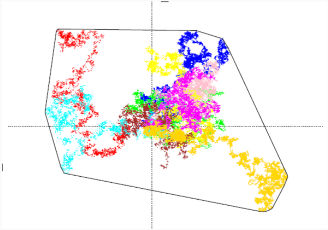

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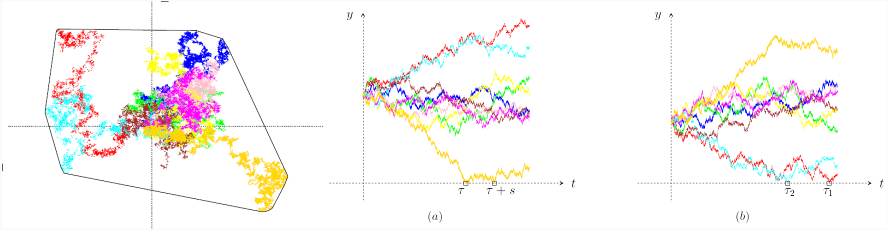

Results (2)

A general formula – for the convex hull *without* the origin

Theorem

For $g : \mathbb{R}^d \to \mathbb{R}^1$ a bounded measurable function, symmetric and invariant with respect to translations,

$$\mathbb{E}\left[g(\mathsf{S}_d)\mathbb{I}_{\{\mathrm{conv}\,\mathsf{S}_d \in \mathcal{F}(\mathcal{C}_d^f)\}}\right] = d!\,\kappa_d\,(2\pi)^{-d/2} \times \mathbb{E}[g(\mathcal{Q}\mathsf{T}_{d-1}) \cdot |\mathrm{conv}\,\mathsf{T}_{d-1}|]$$
$$\times \prod_{l\,:\,k_l \neq 0} \left[\frac{(2(n_l - i_{k_l}^{(l)}) - 1)!!}{(2(n_l - i_{k_l}^{(l)}))!!} \left(i_1^{(l)} (i_2^{(l)} - i_1^{(l)})^3 \ldots (i_{k_l}^{(l)} - i_{k_l-1}^{(l)})^3 \right)^{-1/2} \right]$$
$$\times \int_0^\infty \left\{ \left[\prod_{l\,:\,k_l = 0} p_{n_l}(r) \right] \left[\prod_{l\,:\,k_l \neq 0} q_{i_1^{(l)}}(r) \right] \right.$$
$$\left. + \left[\prod_{l\,:\,k_l = 0} p_{n_l}(-r) \right] \left[\prod_{l\,:\,k_l \neq 0} q_{i_1^{(l)}}(-r) \right] \right\} \exp\left(-\frac{r^2}{2} \sum_{l\,:\,k_l \neq 0} \frac{1}{i_1^{(l)}} \right) \mathrm{d}r.$$

where $\mathsf{T}_{d-1} \sim P_d \mathsf{S}_d$ is a $(d$-1$)$-simplex defined from the same indices as S_d.

Face probability – for the convex hull *without* the origin

- Applying the previous theorem to $g \equiv 1$ leads to:

Theorem

$$\mathbb{P}\left[\mathrm{conv}\,\mathsf{S}_d \in \mathcal{F}(\mathcal{C}_d^f)\right] = d!\,\kappa_d\,(2\pi)^{-d/2} \times \mathbb{E}|\mathrm{conv}\,\mathsf{T}_{d-1}|$$
$$\times \prod_{l\,:\,k_l \neq 0}\left[\frac{(2(n_l - i_{k_l}^{(l)}) - 1)!!}{(2(n_l - i_{k_l}^{(l)}))!!}\left(i_1^{(l)}\,(i_2^{(l)} - i_1^{(l)})^3 \ldots (i_{k_l}^{(l)} - i_{k_l-1}^{(l)})^3\right)^{-1/2}\right]$$
$$\times \int_0^\infty \left\{\left[\prod_{l\,:\,k_l=0} p_{n_l}(r)\right]\left[\prod_{l\,:\,k_l\neq 0} q_{i_1^{(l)}}(r)\right]\right.$$
$$\left. + \left[\prod_{l\,:\,k_l=0} p_{n_l}(-r)\right]\left[\prod_{l\,:\,k_l\neq 0} q_{i_1^{(l)}}(-r)\right]\right\} \exp\left(-\frac{r^2}{2}\sum_{l\,:\,k_l\neq 0}\frac{1}{i_1^{(l)}}\right)\mathrm{d}r.$$

Expected number of faces – for the convex hull *without* the origin

- Summing the previous formula over all choices of k_l's and i_j's leads to:

Theorem

$$\mathbb{E}\,|\mathcal{F}(\mathcal{C}_d^f)| = d!\,\kappa_d\,(2\pi)^{-d/2} \sum_{\substack{k_1,\dots,k_m\ge 0\\ k_1\le n_1,\dots,k_m\le n_m\\ k_1+\dots+k_m=d}} \sum_{\substack{1\le i_1^{(l)}<\dots<i_{k_l}^{(l)}\le n_l\\ l=1,\dots,m\,:\,k_l>0}} \mathbb{E}|\mathrm{conv}\,\mathsf{T}_{d-1}|$$

$$\prod_{l\,:\,k_l\neq 0}\left[\frac{(2(n_l-i_{k_l}^{(l)})-1)!!}{(2(n_l-i_{k_l}^{(l)}))!!}\left(i_1^{(l)}\left(i_2^{(l)}-i_1^{(l)}\right)^3\dots\left(i_{k_l}^{(l)}-i_{k_l-1}^{(l)}\right)^3\right)^{-1/2}\right]$$

$$\times\int_0^\infty\Bigg\{\Bigg[\prod_{l\,:\,k_l=0}p_{n_l}(r)\Bigg]\Bigg[\prod_{l\,:\,k_l\neq 0}q_{i_1^{(l)}}(r)\Bigg]$$

$$+\Bigg[\prod_{l\,:\,k_l=0}p_{n_l}(-r)\Bigg]\Bigg[\prod_{l\,:\,k_l\neq 0}q_{i_1^{(l)}}(-r)\Bigg]\Bigg\}\exp\left(-\frac{r^2}{2}\sum_{l\,:\,k_l\neq 0}\frac{1}{i_1^{(l)}}\right)\mathrm{d}r.$$

Expected boundary surface area – for the convex hull *without* the origin

- Applying the main theorem to $g(\mathsf{S}_d) = |\mathrm{conv}\,\mathsf{S}_d|$, we obtain the expected surface area (i.e. $(d-1)$-dimensional content) of the boundary of the convex hull, $\partial \mathcal{C}_d^f$:

Theorem

$$\mathbb{E}\,|\partial\,\mathcal{C}_d^f| \;=\; d!\,\kappa_d\,(2\pi)^{-d/2} \sum_{\substack{k_1,\dots,k_m \geq 0 \\ k_1 \leq n_1,\dots,k_m \leq n_m \\ k_1+\cdots+k_m = d}} \; \sum_{\substack{1 \leq i_1^{(l)} < \cdots < i_{k_l}^{(l)} \leq n_l \\ l=1,\dots,m\,:\,k_l>0}} \mathbb{E}|\mathrm{conv}\,\mathsf{T}_{d-1}|^2$$

$$\times \prod_{l\,:\,k_l \neq 0} \left[\frac{(2(n_l - i_{k_l}^{(l)}) - 1)!!}{(2(n_l - i_{k_l}^{(l)}))!!} \left(i_1^{(l)} \left(i_2^{(l)} - i_1^{(l)}\right)^3 \dots \left(i_{k_l}^{(l)} - i_{k_l-1}^{(l)}\right)^3 \right)^{-1/2} \right]$$

$$\times \int_0^\infty \Bigg\{ \Bigg[\prod_{l\,:\,k_l=0} p_{n_l}(r) \Bigg] \Bigg[\prod_{l\,:\,k_l \neq 0} q_{i_1^{(l)}}(r) \Bigg]$$

$$+ \Bigg[\prod_{l\,:\,k_l=0} p_{n_l}(-r) \Bigg] \Bigg[\prod_{l\,:\,k_l \neq 0} q_{i_1^{(l)}}(-r) \Bigg] \Bigg\} \exp\left(-\frac{r^2}{2} \sum_{l\,:\,k_l \neq 0} \frac{1}{i_1^{(l)}} \right) \mathrm{d}r.$$

Expected d-dimensional volume – for the convex hull *without* the origin

- Recalling the Cauchy surface area formula: $\mathbb{E}|\mathcal{C}_{d-1}^f| = \frac{\kappa_{d-1}}{d\kappa_d}\mathbb{E}|\partial\,\mathcal{C}_d^f|$ leads to a formula for the expected (d-dimensional) volume of $\mathcal{C}_d^f$

Theorem

$$\mathbb{E}\,|\mathcal{C}_d^f| = d!\,\kappa_d\,(2\pi)^{-(d+1)/2} \sum_{\substack{k_1,\dots,k_m\geq 0\\ k_1\leq n_1,\dots,k_m\leq n_m\\ k_1+\dots+k_m=d+1}} \sum_{\substack{1\leq i_1^{(l)}<\dots<i_{k_l}^{(l)}\leq n_l\\ l=1,\dots,m\,:\,k_l>0}} \mathbb{E}|\mathrm{conv}\,\mathsf{T}_d|^2$$

$$\times \prod_{l\,:\,k_l\neq 0}\left[\frac{(2(n_l-i_{k_l}^{(l)})-1)!!}{(2(n_l-i_{k_l}^{(l)}))!!}\left(i_1^{(l)}(i_2^{(l)}-i_1^{(l)})^3\dots(i_{k_l}^{(l)}-i_{k_l-1}^{(l)})^3\right)^{-1/2}\right]$$

$$\times \int_0^\infty \Bigg\{\Bigg[\prod_{l\,:\,k_l=0} p_{n_l}(r)\Bigg]\Bigg[\prod_{l\,:\,k_l\neq 0} q_{i_1^{(l)}}(r)\Bigg]$$

$$+\Bigg[\prod_{l\,:\,k_l=0} p_{n_l}(-r)\Bigg]\Bigg[\prod_{l\,:\,k_l\neq 0} q_{i_1^{(l)}}(-r)\Bigg]\Bigg\}\exp\left(-\frac{r^2}{2}\sum_{l\,:\,k_l\neq 0}\frac{1}{i_1^{(l)}}\right)\mathrm{d}r.$$

Examples

Single random walk in the plane

- $m = 1$ and $d = 2$
- when $\mathsf{S}_2 = (S_i, S_{i+j})$, $\mathbb{E}|\mathrm{conv}\,\mathsf{T}_1| = \sqrt{2j/\pi}$
- one obtains:

$$\mathbb{E}|\mathcal{F}(\mathcal{C}_2)| = 2!\,\kappa_2\,(2\pi)^{-2/2} \sum_{\substack{k_0,k_1\geq 0\\ k_0+k_1=2}} \sum_{1\leq i_1<\cdots<i_{k_1}\leq n}$$

$$\left[\mathbb{I}_{\{k_0=0\}}\,\frac{2}{i_2-i_1}\,\frac{(2(n-i_2)-1)!!}{(2(n-i_2))!!}\,\frac{(2i_1^{(1)}-1)!!}{(2i_1)!!}\right.$$

$$\left.+\,\mathbb{I}_{\{k_0=1\}}\frac{2}{i_1}\,\frac{(2(n-i_1)-1)!!}{(2(n-i_1))!!}\right]$$

$$=\sum_{j=1}^{n}\frac{2}{j}\sum_{i=0}^{n-j}\frac{(2(n-(i+j))-1)!!}{(2(n-(i+j)))!!}\,\frac{(2i-1)!!}{(2i)!!}$$

$$=2\sum_{j=1}^{n}\frac{1}{j},$$

Single random walk in higher dimension

- $m = 1$ and $d \geq 2$
-

$$\mathbb{E}|\mathcal{F}(\mathcal{C}_d)| = 2 \sum_{j_1=1}^{n+2-d} \sum_{j_2=1}^{n+3-d-j_1} \cdots \sum_{j_{d-1}=1}^{n-(j_1+\cdots+j_{d-2})} \big(j_1 \dots j_{d-1}\big)^{-1}$$

$$\sum_{i=0}^{n-(i+j_1+\cdots+j_{d-1})} \frac{(2(n-(i+j_1+\cdots+j_{d-1}))-1)!!}{(2(n-(j_1+\cdots+j_{d-1})))!!} \frac{(2i-1)!!}{(2i)!!}$$

$$= 2 \sum_{j_1=1}^{n+2-d} \sum_{j_2=1}^{n+3-d-j_1} \cdots \sum_{j_{d-1}=1}^{n-(j_1+\cdots+j_{d-2})} \big(j_1 \dots j_{d-1}\big)^{-1}$$

$$= 2 \sum_{\substack{j_1+\cdots+j_{d-1}\leq n \\ j_1,\dots,j_{d-1}\geq 1}} \frac{1}{j_1 \cdot \ldots \cdot j_{d-1}},$$

Independent Gaussian points

- $\forall l \leq m,\ n_l = 1$

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- $\forall l \leq m,\ n_l = 1$
- Without the origin: standard Gaussian polytope
- $$\mathbb{E}|\mathcal{C}_d^f| = \frac{\kappa_d\, m!}{d!\,(m-d-1)!} \int_{-\infty}^{+\infty} \Phi^{m-d-1}(r)\, \varphi^{d+1}(r) \mathrm{d}r,$$

  which is indeed Efron's formula.

Independent Gaussian points

- $\forall l \leq m,\ n_l = 1$
- With the origin: Gaussian polytope *with* 0
- Note that
  $\mathbb{E}_{k_0=1}|\mathrm{conv}\,\mathsf{T}_d|^2 = 1/d!$ whereas $\mathbb{E}_{k_0=0}|\mathrm{conv}\,\mathsf{T}_d|^2 = (d+1)/d!$ (Miles, 1971)

Independent Gaussian points

- $\forall l \leq m,\ n_l = 1$
- With the origin: Gaussian polytope *with* 0
- $$\mathbb{E}|\mathcal{C}_d| \;=\; \frac{\binom{m}{d}}{2^{m-\frac{d}{2}}\Gamma\left(\frac{d}{2}+1\right)} + \frac{\kappa_d\, m!}{d!\,(m-d-1)!}\int_0^\infty \Phi^{m-d-1}(r)\,\varphi^{d+1}(r)\mathrm{d}r,$$

  in full agreement with the formula established by Kabluchko and Zaporozhets.

Conclusion

Ongoing work...

- general formulae (also for Lévy processes)
- new results
- extensions?

R-F & Zaporozhets (~~2020~~ 2021). *Preprint available*

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Thank you for your attention!