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ON DECAY-SURGE POPULATION MODELS

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Abstract

We consider continuous space-time decay-surge population models which are semi-stochastic processes for which deterministically declining populations, bound to fade away, are reinvigorated at random times by bursts or surges of random sizes. In a particular separable framework (in a sense made precise below) we provide explicit formulae for the scale (or harmonic) function and the speed measure of the process. The behavior of the scale function at infinity allows to formulate conditions under which such processes either explode or are transient at infinity, or Harris recurrent. A description of the structures of both the discrete-time embedded chain and extreme record chain of such continuous-time processes is supplied.

Keywords: declining population models, surge processes, PDMP, Hawkes processes, scale function, Harris recurrence.

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1. Introduction

This paper deals with decay-surge population models where a deterministically declining evolution following some nonlinear flow is interrupted by bursts of random

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sizes occurring at random times. Decay-surge models are natural models of many physical and biological phenomena, including the evolution of ageing and declining populations which are reinvigorated by immigration, the height of the membrane potential of a neuron decreasing in between successive spikes of its presynaptic neurons due to leakage effects and jumping upwards after the action potential emission of its presynaptic partners, work processes in single-server queueing systems as from Cohen (1982), etc. Our preferred physical image will be the one of ageing populations subject to immigration.

Decay-surge models have been extensively studied in the literature, see among others Eliazar and Klafter (2006), and Harrison and Resnick (1976) and (1978). Most studies concentrate however on non-Markovian models such as shot noise or Hawkes processes, where superpositions of overlapping decaying populations are considered, see Brémaud and Massoulié (1996), Eliazar and Klafter (2007) and (2009), Huillet (2021), Kella and Stadje (2001), Kella (2009) and Resnick and Tweedie (1982).

Inspired by storage processes for dams, the papers of Resnick and Tweedie (1982), Kella (2009), Çinlar and Pinsky (1971), Asmussen and Kella (1996), Boxma et al. (2011), Boxma at al. (2006), Harrison and Resnick (1976), Brockwell et al. (1982) are mostly concerned with growth-collapse models when growth is from *stochastic* additive inputs such as compound Poisson or Lévy processes or renewal processes. Here, the water level of a dam decreasing deterministically according to some fixed water release program is subject to sudden uprises due to rain or flood. Growth-collapse models are also very relevant in the Burridge-Knopoff stress-release model of earthquakes and continental drift, as from Carlson et al. (1994), and in stick-slip models of interfacial friction, as from Richetti et al. (2001). As we shall see, growth-collapse models are in some sense 'dual' to decay-surge models.

In contrast with these last papers, and as in the works of Eliazar and Klafter (2006), (2007) and (2009), we concentrate in the present work on a *deterministic* and continuous decay motion in between successive surges, described by a nonlinear flow, determining the decay rate of the population and given by

$$x_t(x) = x - \int_0^t \alpha(x_s(x))ds,\ t \geq 0,\ x_0(x) = x \geq 0.$$

In our process, upward jumps (surges) occur with state dependent rate $\beta(x)$, when the current state of the process is x. When a jump occurs, the present size of the population x is replaced by a new random value $Y(x) > x$, distributed according to some transition kernel $K(x, dy), y \geq x$.

This leads to the study of a quite general family of continuous-time piecewise deterministic Markov processes (PDMP's) $X_t(x)$ representing the size of the population at time t when started from the initial value $x \geq 0$. See Davis (1984). The infinitesimal generator of this process is given for smooth test functions by

$$\mathcal{G}u(x) = -\alpha(x)u'(x) + \beta(x) \int_{(x,\infty)} K(x, dy)[u(y) - u(x)] \ , \ x \geq 0,$$

under suitable conditions on the parameters α, β and $K(x, dy)$ of the process. In the sequel we focus on the study of separable kernels $K(x, dy)$ where for each $0 \leq x \leq y$,

$$\int_{(y,\infty)} K(x, dz) = \frac{k(y)}{k(x)}, \tag{1}$$

for some positive non-increasing function $k : [0, \infty) \to [0, \infty]$ which is continuous on $(0, \infty)$.

The present paper proposes a precise characterization of the probabilistic properties of the above process in this separable frame. Supposing that $\alpha(x)$ and $\beta(x)$ are continuous and positive on $(0, \infty)$, the main ingredient of our study is the function

$$\Gamma(x) = \int_1^x \gamma(y)dy, \text{ where } \gamma(y) = \beta(y)/\alpha(y), \ y, x \geq 0. \tag{2}$$

Supposing that $\Gamma(\cdot)$ is a **space transform,** that is, $\Gamma(0) = -\infty$ and $\Gamma(\infty) = \infty$, we show that

1. Starting from some strictly positive initial value $x > 0$, the process does not get extinct (does not hit 0) in finite time almost surely (Proposition 2). In particular, imposing additionally $k(0) < \infty$, we can study the process in restriction to the state space $(0, \infty)$. This is what we do in the sequel.

2. The function
$$s(x) = \int_1^x \gamma(y)e^{-\Gamma(y)}/k(y)dy, \ x \geq 0, \tag{3}$$
is a **scale function** of the process, that is, solves $\mathcal{G}s(x) = 0$ (Proposition 3). It is always strictly increasing and satisfies $s(0) = -\infty$ under our assumptions. But it might not be a space transform, that is, $s(\infty)$ can take finite values.

3. This scale function plays a key role in the understanding of the exit probabilities of the process and yields conditions under which the process either explodes in finite time or is transient at infinity. More precisely, if $s(\infty) < \infty$, we have the following explicit formula for the exit probabilities. For any $0 < a < x < b$,

$$\mathbb{P}(X_t \text{ enters } (0,a] \text{ before entering } [b,\infty) \,|X_0 = x) = \frac{s(x) - s(a)}{s(b) - s(a)}. \tag{4}$$

   (Proposition 4).

   Taking $b \to \infty$ in the above formula, we deduce from this that $s(\infty) < \infty$ implies either that the process explodes in finite time (possesses an infinite number of jumps within some finite time interval) or that it is transient at infinity.

   Due to the asymmetric dynamic of the process (continuous motion downwards and jumps upwards such that entering the interval $[b, \infty)$ starting from $x < b$ always happens by a jump), (4) does not hold if $s(\infty) = \infty$.

4. Imposing additionally that $\beta(0) > 0$, **Harris recurrence (positive or null)** of the process is equivalent to the fact that s is a space transform, that is, $s(\infty) = \infty$ (Theorem 2).

   In this case, up to constant multiples, the unique invariant measure of the process possesses a Lebesgue density (speed density) given by

$$\pi(x) = \frac{k(x)e^{\Gamma(x)}}{\alpha(x)}, x > 0.$$

   More precisely, we show how the scale function can be used to obtain Foster-Lyapunov criteria in the spirit of Meyn and Tweedie (1993) implying the non-explosion of the process together with its recurrence under additional irreducibility properties. Additional conditions, making use of the speed measure, under which first moments of hitting times are finite, are also supplied in this setup.

Organization of the paper. In Section 2, we introduce our model and state some first results. Most importantly, we establish a simple relationship between decay-surge models and growth-collapse models as studied in Goncalves, Huillet and Löcherbach (2022) that allows us to obtain explicit representations of the law of the first jump time and of the associated speed measure without any further study. Section 3 is devoted to the proof of the existence of the scale function (Proposition 3) together with the study of first moments of hitting times which are shown to be finite if the speed density is integrable at $+\infty$ (Proposition 6). Section 4 then collects our main results. If the scale function is a space transform, it can be naturally transformed into a Lyapunov function in the sense of Meyn and Tweedie (1993) such that the process does not explode in finite time and comes back to certain compact sets infinitely often (Proposition 7). Using the regularity produced by the jump heights according to the absolutely continuous transition kernel $K(x, dy)$, Theorem 1 establishes then a local Doeblin lower bound on the transition operator of the process - a key ingredient to prove Harris recurrence which is our main result Theorem 2. Several examples are supplied including one related to linear Hawkes processes and to shot-noise processes. In a last part of the work, we focus on the embedded chain of the process, sampled at the jump times, which, in addition to its fundamental relevance, is easily amenable to simulations. Following Adke (1993), we also draw the attention to the structure of the extreme record chain of $X_t(x)$, allowing in particular to derive the distribution of the first upper record time and overshoot value, as a level crossing time and value. This study is motivated by the understanding of the time of the first crossing of some high population level and the amount of the corresponding overshoot, as, besides extinction, populations can face overcrowding.

2. The model, some first results and a useful duality property

We study population decay models with random surges described by a Piecewise Deterministic Markov Process (PDMP) $X_t, t \geq 0$, starting from some initial value $x \geq 0$ at time 0 and taking values in $[0, \infty)$. The main ingredients of our model are

1. The drift function $\alpha(x)$. We suppose that $\alpha : [0, \infty) \to [0, \infty)$ is continuous, with $\alpha(x) > 0$ for all $x > 0$. In between successive jumps, the process follows the decaying dynamic
$$\dot{x}_t(x) = -\alpha\left(x_t(x)\right), x_0(x) = x \geq 0. \tag{5}$$
2. The jump rate function $\beta(x)$. We suppose that $\beta : (0, \infty) \to [0, \infty)$ is continuous and $\beta(x) > 0$ for all $x > 0$.

3. The jump kernel $K(x, dy)$. This is a transition kernel from $[0, \infty)$ to $[0, \infty)$ such that for any $x > 0$, $K(x, [x, \infty)) = 1$. Writing $K(x, y) = \int_{(y,\infty)} K(x, dz)$, we suppose that $K(x, y)$ is jointly continuous in x and y.

In between successive jumps, the population size follows the deterministic flow $x_t(x)$ given in (5). For any $0 \leq a < x$, the integral

$$t_a(x) := \int_a^x \frac{dy}{\alpha(y)} \tag{6}$$

is the time for the flow to hit a starting from x. In particular, starting from $x > 0$, the flow reaches 0 after some time $t_0(x) = \int_0^x \frac{dy}{\alpha(y)} \leq \infty$. We refer to [16], Section 2, for a variety of examples of such decaying flows that can hit zero in finite time or not.

Jumps occur at state dependent rate $\beta(x)$. At the jump times, the size of the population grows by a random amount $\Delta(X_{t-}) > 0$ of its current size X_{t-}. Writing $Y(X_{t-}) := X_{t-} + \Delta(X_{t-})$ for the position of the process right after its jump, $Y(X_{t-})$ is distributed according to $K(X_{t-}, dy)$.

Up to the next jump time, X_t then decays again, following the deterministic dynamics (5), started at the new value $Y(X_{t-}) := X_{t-} + \Delta(X_{t-})$.

We are thus led to consider the PDMP X_t with state-space $[0, \infty)$ solving

$$dX_t = -\alpha(X_t)\, dt + \Delta(X_{t-}) \int_0^\infty \mathbf{1}_{\{r \leq \beta(X_{t-}(x))\}} M(dt, dr), \; X_0 = x, \tag{7}$$

where $M(dt, dr)$ is a Poisson measure on $[0, \infty) \times [0, \infty)$. Taking $dt \ll 1$ be the system's scale, this dynamics means alternatively that we have transitions

$$\begin{aligned} X_{t-} &= x \to x - \alpha(x)\, dt \text{ with probability } 1 - \beta(x)\, dt \\ X_{t-} &= x \to x + \Delta(x) \text{ with probability } \beta(x)\, dt. \end{aligned}$$

It is a nonlinear version of the Langevin equation with jumps.

2.1. Discussion of the jump kernel

We have

$$\mathbb{P}(Y(x) > y \mid X_{t-} = x) = K(x, y) = \int_{(y,\infty)} K(x, dz).$$

Clearly $K(x, y)$ is a non-increasing function of y for all $y \geq x$ satisfying $K(x, y) = 1$ for all $y < x$. By continuity, this implies that $K(x, x) = 1$ such that the law of $\Delta(x)$ has no atom at 0.

In the sequel we concentrate on the separable case

$$K(x,y) = \frac{k(y)}{k(x)}, \tag{8}$$

where $k : [0, \infty) \to [0, \infty]$ is any positive non-increasing function. In what follows we suppose that k is continuous and finite on $(0, \infty)$.

Fix $z > 0$ and assume $y = x + z$. Then

$$\mathbf{P}(Y(x) > y) = \frac{k(x+z)}{k(x)}.$$

Depending on $k(x)$, this probability can be a decreasing or an increasing function of x for each z.

Example 1. Suppose $k(x) = e^{-x^{\alpha}}$, $\alpha > 0,\ x \geq 0$ (a Weibull distribution).

- If $\alpha < 1$, then $\partial_x K(x, x+z) > 0$, so that the larger x, the larger $\mathbf{P}(Y(x) > x+z)$. In other words, if the population stays high, the probability of a large number of immigrants will be enhanced. There is a positive feedback of x on $\Delta(x)$, translating a herd effect.

- If $\alpha = 1$, then $\partial_x K(x, x+z) = 0$ and there is no feedback of x on the number of immigrants, which is then exponentially distributed.

- If $\alpha > 1$, then $\partial_x K(x, x+z) < 0$ and the larger x, the smaller the probability $\mathbf{P}(Y(x) > x+z)$. In other words, if the population stays high, the probability of a large number of immigrants will be reduced. There is a negative feedback of x on $\Delta(x)$.

Example 2. The case $k(0) < \infty$. Without loss of generality, we may take $k(0) = 1$. Assume that $k(x) = \mathbf{P}(Z > x)$ for some proper random variable $Z > 0$ and that

$$Y(x) \stackrel{d}{=} Z \mid Z > x$$

so that $Y(x)$ is amenable to the truncation of Z above x. Thus

$$\mathbf{P}(Y(X_{t-}) > y \mid X_{t-} = x) = \frac{\mathbf{P}(Z > y, Z > x)}{\mathbf{P}(Z > x)} = \frac{k(y)}{k(x)}, \text{ for } y > x.$$

A particular (exponential) choice is

$$k(x) = e^{-\theta x},\ \theta > 0,$$

with $\mathbf{P}\left(Y\left(X_{t-}\right)>y \mid X_{t-}=x\right)=e^{-\theta(y-x)}$ depending only on $y-x$. Another possible one is (Pareto): $k(x)=(1+x)^{-c}$, $c>0$.

Note $K(0,y)=k(y)>0$ for all $y>0$, and $k(y)$ turns out to be the cpdf of a jump above y, starting from 0: state 0 is **reflecting**.

The case $k(0)=\infty$. Consider $k(x)=\int_x^\infty \mu(Z\in dy)$ for some positive Radon measure μ with infinite total mass. In this case,

$$\mathbf{P}\left(Y\left(X_{t-}\right)>y \mid X_{t-}=x\right)=\frac{\mu(Z>y,Z>x)}{\mu(Z>x)}=\frac{k(y)}{k(x)},\ \text{for } y>x.$$

Now, $K(0,y)=0$ for all $y>0$ and state 0 becomes **attracting**. An example is $k(x)=x^{-c}$, $c>0$, which is not a cpdf.

The ratio $k(y)/k(x)$ is thus the conditional probability that a jump is greater than the level y given that it did occur and that it is greater than the level x, see Eq. (1) of Eliazar and Klafter (2009) for a similar choice.

Our motivation for choosing the separable form $K(x,y)=k(y)/k(x)$ is that it accounts for the possibility of having state 0 either absorbing or reflecting for upwards jumps launched from 0 and also that it can account for both a negative or positive feedback of the current population size on the number of incoming immigrants.

Example 3. One can think of many other important and natural choices of $K(x,y)$, not in the separable class, among which those for which

$$K(x,dy)=\delta_{Vx}(dy)$$

for some random variable $V>1$. For this class of kernels, state 0 is **always** attracting. For example, choosing $V=1+E$ where:

1/ E is exponentially distributed with pdf $e^{-\theta x}$, $Y(x)=Vx$ yields

$$\mathbf{P}\left(Y(x)>y \mid X_{-}=x\right)=K(x,y)=\mathbf{P}\left((1+E)x>y\right)=e^{-\theta\left(\frac{y}{x}-1\right)},$$

2/ E is Pareto distributed with pdf $(1+x)^{-c}$, $c>0$, $Y(x)=Vx$ yields

$$\mathbf{P}\left(Y(x)>y \mid X_{-}=x\right)=K(x,y)=\mathbf{P}\left((1+E)x>y\right)=(y/x)^{-c},$$

both with $K(0,y)=0$. When

3/: $V\sim\delta_v$, $v>1$, then $K(x,y)=\mathbf{1}(y/x\leq v)$. The three kernels depend on y/x.

Note that in all three cases, $\partial_x K(x, x+z) > 0$, so that the larger x, the larger $\mathbf{P}(Y(x) > x+z)$. If the population stays high, the probability of a large number of immigrants will be enhanced. There is a positive feedback of x on $\Delta(x)$, translating a herd effect.

Remark 1. A consequence of the separability condition of K is the following. Consider a Markov sequence of after-jump positions defined recursively by $Z_n = Y(Z_{n-1})$, $Z_0 = x_0$. With $x_m > x_{m-1}$, we have

$$\mathbf{P}(Z_m > x_m \mid Z_{m-1} = x_{m-1}) = K(x_{m-1}, x_m), \; m = 1, ..., n,$$

so that, with $x_0 < x_1 < ... < x_n$, and under the separability condition on K, the product

$$\prod_{m=1}^{n} \mathbf{P}(Z_m > x_m \mid Z_{m-1} = x_{m-1}) = \prod_{m=1}^{n} K(x_{m-1}, x_m) = \prod_{m=1}^{n} \frac{k(x_m)}{k(x_{m-1})} = \frac{k(x_n)}{k(x_0)}$$

only depends on the initial and terminal states (x_0, x_n) and not on the full path $(x_0, ..., x_n)$.

2.2. The infinitesimal generator

In what follows we always work with separable kernels. Moreover, we write $X_t(x)$ for the process given in (7) to emphasise the dependence on the starting point x, that is, $X_t(x)$ designs the process with the above dynamics (7) and satisfying $X_0(x) = x$. If the value of the starting point x is not important, we shall also write X_t instead of $X_t(x)$.

Under the separability condition, the infinitesimal generator of X_t acting on bounded smooth test functions u takes the following simple form

$$(\mathcal{G}u)(x) = -\alpha(x) u'(x) + \frac{\beta(x)}{k(x)} \int_x^\infty k(y) u'(y) \, dy, x \geq 0. \tag{9}$$

Remark 2. In Eliazar and Klafter (2006), a particular scale-free version of decay-surge models with $\alpha(x) \propto x^a$, $\beta(x) \propto x^b$ and $k(x) \propto x^{-c}$, $c > 0$, has been investigated.

Remark 3. If x_t goes extinct in finite time $t_0(x) < \infty$, since x_t is supposed to represent the size of some population, we need to impose $x_t = 0$ for $t \geq t_0(x)$, forcing state 0 to be absorbing. From this time on, X_t can re-enter the positive orthant if there is a positive probability to move from 0 to a positive state meaning $k(0) < \infty$ and $\beta(0) > 0$. In such a case, the first time X_t hits state 0 is only a first local extinction time the expected value of which needs to be estimated. The question of the time elapsed between consecutive local extinction times (excursions) also arises.

On the contrary, for situations for which $k(0) = \infty$ or $\beta(0) = 0$, the first time X_t hits state 0 will be a global extinction time.

2.3. Relation between decay-surge and growth-collapse processes

In this subsection, we exhibit a natural relationship between decay-surge population models, as studied here, and growth-collapse models as developed in Boxma et al. (2006), Goncalves et al. (2022), Gripenberg (1983) and Hanson and Tuckwell (1978). Growth-collapse models describe deterministic population growth where at random jump times the size of the population undergoes a catastrophe reducing its current size to a random fraction of it. More precisely, the generator of a growth-collapse process, having parameters $(\tilde{\alpha}, \tilde{\beta}, \tilde{h})$, is given for all smooth test functions by

$$(\tilde{\mathcal{G}}u)(x) = \tilde{\alpha}(x)u'(x) - \tilde{\beta}(x)/\tilde{h}(x)\int_0^x u'(y)\tilde{h}(y)dy, x \geq 0. \tag{10}$$

In the above formula, $\tilde{\alpha}, \tilde{\beta}$ are continuous and positive functions on $(0, \infty)$, and $\tilde{h}$ is positive and non-decreasing on $(0, \infty)$.

In what follows, consider a decay-surge process X_t defined by the triple (α, β, k) and let $\widetilde{X}_t = 1/X_t$.

Proposition 1. *The process $\widetilde{X}_t$ is a growth-collapse process as studied in [15] with triple $\left(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{h}\right)$ given by*

$$\widetilde{\alpha}(x) = x^2\alpha(1/x), \widetilde{\beta}(x) = \beta(1/x) \text{ and } \widetilde{h}(x) = k(1/x),\ x > 0.$$

Proof. Let u be any smooth test function and study $u(\tilde{X}_t) = u \circ g(X_t)$ with $g(x) = 1/x$. By Ito's formula for processes with jumps,

$$u(\tilde{X}_t) = u \circ g(X_t) = u(\tilde{X}_0) + \int_0^t \mathcal{G}(u \circ g)(X_{t'})dt' + M_t,$$

where M_t is a local martingale. We obtain

$$\begin{aligned} \mathcal{G}(u \circ g)(x) &= -\alpha(x)(u \circ g)'(x) + \frac{\beta(x)}{k(x)}\int_x^\infty (u \circ g)'(y)k(y)dy \\ &= \frac{1}{x^2}\alpha(x)u'(\frac{1}{x}) + \frac{\beta(x)}{k(x)}\int_x^\infty u'(\frac{1}{y})k(y)\frac{-dy}{y^2} \end{aligned}$$

Using the change of variable $y = 1/x$, this last expression can be rewritten as

$$\frac{1}{x^2}\alpha(x)u'(\frac{1}{x}) - \frac{\beta(x)}{k(x)}\int_0^{1/x} u'(z)k(\frac{1}{z})dz = \widetilde{\alpha}(y)u'(y) - \frac{\widetilde{\beta}(y)}{\widetilde{h}(y)}\int_0^{y} u'(t)\widetilde{h}(t)dt$$

which is the generator of the process $\widetilde{X}_t$. □

In what follows we speak about the above relation between the decay-surge (DS) process X and the growth-collapse (GC) process $\tilde{X}$ as *DS-GC-duality*. Some simple properties of the process X follow directly from the above duality as we show next. Of course, the above duality does only hold up to the first time one of the two processes leaves the interior $(0, \infty)$ of its state space. Therefore a particular attention has to be paid to state 0 for X_t or equivalently to state $+\infty$ for $\tilde{X}_t$. Most of our results will only hold true under conditions ensuring that, starting from $x > 0$, the process X_t will not hit 0 in finite time.

Another important difference between the two processes is that the simple transformation $x \mapsto 1/x$ maps a priori unbounded sample paths X_t into bounded ones $\tilde{X}_t$ (starting from $\tilde{X}_0 = 1/x$, almost surely, $\tilde{X}_t \leq \tilde{x}_t(1/x)-$ a relation which does not hold for X).

2.4. First consequences of the DS-GC-duality

Given $X_0 = x > 0$, the first jump times both of the DS-process X_t starting from x, and of the GC-process $\tilde{X}_t$, starting from $1/x$, coincide and are given by

$$T_x = \inf\{t > 0 : X_t \neq X_{t-} | X_0 = x\} = \tilde{T}_{\frac{1}{x}} = \inf\{t > 0 : \tilde{X}_t \neq \tilde{X}_{t-} | \tilde{X}_0 = \frac{1}{x}\}.$$

Introducing

$$\Gamma(x) := \int_1^x \gamma(y)\, dy, \text{ where } \gamma(x) := \beta(x)/\alpha(x), x > 0,$$

and the corresponding quantity associated to the process $\tilde{X}_t$,

$$\widetilde{\Gamma}(x) = \int_1^x \tilde{\gamma}(y)\, dy, \; \tilde{\gamma}(x) = \tilde{\beta}(x)/\tilde{\alpha}(x), x > 0,$$

clearly, $\widetilde{\Gamma}(x) = -\Gamma(1/x)$ for all $x > 0$.

Arguing as in Sections 2.4 and 2.5 of [15], a direct consequence of the above duality is the fact that for all $t < t_0(x)$,

$$\mathbb{P}(T_x > t) = e^{-\int_0^t \beta(x_s(x))ds} = e^{-[\Gamma(x) - \Gamma(x_t(x))]}. \tag{11}$$

To ensure that $\mathbb{P}(T_x < \infty) = 1$, in accordance with Assumption 1 of [15] we will impose the condition

Assumption 1. $\Gamma(0) = -\infty$.

Proposition 2. *Under Assumption 1, the stochastic process* $X_t(x), x > 0$, *necessarily jumps before reaching* 0. *In particular, for any* $x > 0$, $X_t(x)$ *almost surely never reaches* 0 *in finite time.*

Proof. By duality, we have

$$\mathbb{P}(X \text{ jumps before reaching } 0|X_0 = x) = \mathbb{P}(\tilde{X} \text{ jumps before reaching } +\infty|\tilde{X}_0 = \frac{1}{x}) = 1,$$

as has been shown in Section 2.5 of [15], and this implies the assertion. □

In particular, the only situation where the question of the extinction of the process X makes sense (either local or total) is when $t_0(x) < \infty$ and $\Gamma(0) > -\infty$.

Example 4. We give an example where finite time extinction of the process is possible. Suppose $\alpha(x) = \alpha_1 x^a$ with $\alpha_1 > 0$ and $a < 1$. Then $x_t(x)$, started at x, hits 0 in finite time $t_0(x) = x^{1-a}/[\alpha_1(1-a)]$, with

$$x_t(x) = \left(x^{1-a} + \alpha_1(a-1)t\right)^{1/(1-a)},$$

see [16]. Suppose $\beta(x) = \beta_1 > 0$, constant. Then, with $\gamma_1 = \beta_1/\alpha_1 > 0$

$$\Gamma(x) = \int_1^x \gamma(y)\,dy = \frac{\gamma_1}{1-a}\left(x^{1-a} - 1\right)$$

with $\Gamma(0) = -\frac{\gamma_1}{1-a} > -\infty$. Assumption 1 is not fulfilled, so X can hit 0 in finite time and there is a positive probability that $T_x = \infty$. On this last event, the flow $x_t(x)$ has all the time necessary to first hit 0 and, if in addition the kernel k is chosen so as $k(0) = \infty$, to go extinct definitively. The time of extinction $\tau(x)$ of X itself can be deduced from the renewal equation in distribution

$$\tau(x) \overset{d}{=} t_0(x)\,\mathbf{1}_{\{T_x=\infty\}} + \tau'\left(Y\left(x_{T_x}(x)\right)\right)\mathbf{1}_{\{T_x<\infty\}}$$

where τ' is a copy of τ.

We conclude that for this family of models, X itself goes extinct in finite time. This is an interesting regime that we shall not investigate any further.

Let us come back to the discussion of Assumption 1. It follows immediately from Eq. (13) in [15] that for $x > 0$, under Assumption 1 and supposing that $t_0(x) = \infty$ for all $x > 0$,

$$\mathbb{E}\left(T_x\right) = e^{-\Gamma(x)} \int_0^x \frac{dz}{\alpha\left(z\right)} e^{\Gamma(z)}.$$

Clearly, when $x \to 0$, $\mathbb{E}\left(T_x\right) \sim 1/\alpha\left(x\right)$.

Remark 4. (i) If $\beta\left(0\right) > 0$ then Assumption 1 implies $t_0(x) = \infty$, such that 0 is not accessible.

(ii) Notice also that $t_0(x) < \infty$ together with Assumption 1 implies that $\beta(0) = \infty$ such that the process $X_t(x)$ is prevented from hitting 0 even though $x_t(x)$ reaches it in finite time due to the fact that the jump rate $\beta(x)$ blows up as $x \to 0$.

Remark 5. The DS-GC-duality makes it possible to translate known results on moments for growth-collapse models obtained e.g. in [11], [29] or [30], to obtain analogous moment results for decay-surge models.

2.5. Classification of state 0

Recall that for all $x > 0$,

$$t_0(x) = \int_0^x \frac{dy}{\alpha\left(y\right)}$$

represents the time required for x_t to move from $x > 0$ to 0. So:

$$\begin{aligned} &\text{If } t_0(x) < \infty \text{ and } \Gamma(0) > -\infty, && \text{state 0 is accessible.} \\ &\text{If } t_0(x) = \infty \text{ or } \Gamma(0) = -\infty, && \text{state 0 is inaccessible.} \end{aligned}$$

We therefore introduce the following conditions which apply in the separable case $K(x, y) = k(y)/k(x)$.

Condition (R): $\beta\left(0\right) > 0$ and $K\left(0, y\right) = k\left(y\right)/k\left(0\right) > 0$ for some $y > 0$.

Condition (A): $\frac{\beta(0)}{k(0)} k(y) = 0$ for all $y > 0$.

State 0 is reflecting if condition (R) is satisfied and it is absorbing if condition (A) is satisfied.

This leads to four possible combinations for the boundary state 0:

1. **Condition (R)** is satisfied, and $t_0(x) < \infty$ and $\Gamma(0) > -\infty$: regular (reflecting and accessible).

2. **Condition (R)** is satisfied, and $t_0(x) = \infty$ or $\Gamma(0) = -\infty$: entrance (reflecting and inaccessible).

3. **Condition (A)** is satisfied, and $t_0(x) < \infty$ and $\Gamma(0) > -\infty$: exit (absorbing and accessible).

4. **Condition (A)** is satisfied, and $t_0(x) = \infty$ or $\Gamma(0) = -\infty$: natural (absorbing and inaccessible).

2.6. Speed measure.

Suppose now an invariant measure (or speed measure) $\pi(dy)$ exists. Since we supposed $\alpha(x) > 0$ for all $x > 0$, we necessarily have $x_\infty(x) = 0$ for all $x > 0$ and so the support of π is $[x_\infty(x) = 0, \infty)$. Thanks to our duality relation, by Eq. (19) of [15] the explicit expression of the speed measure is given by $\pi(dy) = \pi(y)dy$ with

$$\pi(y) = C\frac{k(y)\, e^{\Gamma(y)}}{\alpha(y)}, \tag{12}$$

up to a multiplicative constant $C > 0$. If and only if this function is integrable at 0 and ∞, $\pi(y)$ can be tuned to a probability density function.

Remark 6. (i)When $k(x) = e^{-\kappa_1 x}$, $\kappa_1 > 0$, $\alpha(x) = \alpha_1 x$ and $\beta(x) = \beta_1 > 0$ constant, $\Gamma(y) = \gamma_1 \log y$, $\gamma_1 = \beta_1/\alpha_1$ and

$$\pi(y) = Cy^{\gamma_1 - 1}e^{-\kappa_1 y}$$

a Gamma(γ_1, κ_1) distribution. This result is well-known, corresponding to the linear decay-surge model (a jump version of the damped Langevin equation) having an invariant (integrable) probability density, see Malrieu (2015). We shall show later that the corresponding process X is positive recurrent.

(ii)A less obvious power-law example is as follows: Assume $\alpha(x) = \alpha_1 x^a$ $(a > 1)$ and $\beta(x) = \beta_1 x^b$, $\alpha_1, \beta_1 > 0$ so that $\Gamma(y) = \frac{\gamma_1}{b-a+1}y^{b-a+1}$. We have $\Gamma(0) = -\infty$ if we assume $b - a + 1 = -\theta$ with $\theta > 0$, hence $\Gamma(y) = -\frac{\gamma_1}{\theta}y^{-\theta}$. Taking $k(y) = e^{-\kappa_1 y^\eta}$, $\kappa_1, \eta > 0$

$$\pi(y) = Cy^{-a}e^{-\left(\kappa_1 y^\eta + \frac{\gamma_1}{\theta}y^{-\theta}\right)}$$

which is integrable both at $y = 0$ and $y = \infty$. As a special case, if $a = 2$ and $b = 0$

(constant jump rate $\beta(x)$), $\eta = 1$

$$\pi(y) = Cy^{-2} e^{-\left(\kappa_1 y + \gamma_1 y^{-1}\right)},$$

an inverse Gaussian density.

(iii) In Eliazar and Klafter (2007) and (2009), a special case of our model was introduced for which $k(y) = \beta(y)$. In such cases,

$$\pi(y) = C\gamma(y) e^{\Gamma(y)}$$

so that

$$\int_0^x \pi(y)\, dy = C\left(e^{\Gamma(x)} - e^{\Gamma(0)}\right) = Ce^{\Gamma(x)},$$

under the Assumption $\Gamma(0) = -\infty$. If in addition $\Gamma(\infty) < \infty$, $\pi(y)$ can be tuned to a probability density.

3. Scale function and hitting times

In this section we start by studying the scale function of X_t, before switching to hitting times features that make use of it. A scale function $s(x)$ of the process is any function solving $(\mathcal{G}s)(x) = 0$. In other words, a scale function is a function that transforms the process into a local martingale. Of course, any constant function is solution. Notice that for the growth-collapse model considered in [15], other scale functions than the constant ones do not exist.

In what follows we are interested in non-constant solutions and conditions ensuring the existence of those. To clarify ideas, we introduce the following condition

Assumption 2. *Let*

$$s(x) = \int_1^x \gamma(y) e^{-\Gamma(y)}/k(y) dy,\ x \geq 0, \tag{13}$$

and suppose that $s(\infty) = \infty$.

Notice that Assumption 2 implies that $k(\infty) = 0$ which is reasonable since it prevents jumps of the process X_t jumping from some finite position X_{t-} to an after jump position $X_t = X_{t-} + \Delta(X_{t-}) = +\infty$.

Proposition 3. *(1) Suppose* $\Gamma(\infty) = \infty$. *Then the function* s *introduced in* (13) *above is a strictly increasing version of the scale function of the process obeying* $s(1) = 0$.

(1.1) If additionally Assumptions 1 and 2 hold and if $k(0) < \infty$, *then* $s(0) = -\infty$ *and* $s(\infty) = \infty$, *such that* s *is a space transform* $[0, \infty) \to [-\infty, \infty)$.

(1.2) If Assumption 2 does not hold, then

$$s_1(x) = \int_x^\infty \gamma(y) e^{-\Gamma(y)}/k(y) dy = s(\infty) - s(x)$$

is a version of the scale function which is strictly decreasing, positive, such that $s_1(\infty) = 0$.

(2) Finally, suppose that $\Gamma(\infty) < \infty$. *Then the only scale functions belonging to* C^1 *are the constant ones.*

Remark 7. 1. We shall see later that – as in the case of one-dimensional diffusions, see e.g. Example 2 in Section 3.8 of Has'minskii (1980) – the fact that s is a space transform as in item (1.1) above is related to the Harris recurrence of the process.

2. The assumption $\Gamma(\infty) < \infty$ of item (3) above corresponds to Assumption 2 of [15] where this was the only case that we considered. As a consequence, for the GC-model considered there we did not dispose of non-constant scale functions.

Proof. To find a C^1-scale function s, it necessarily solves

$$(\mathcal{G}s)(x) = -\alpha(x) s'(x) + \beta(x)/k(x) \int_x^\infty k(y) s'(y) dy = 0$$

such that for all $x > 0$,

$$k(x) s'(x) - \gamma(x) \int_x^\infty k(y) s'(y) dy = 0. \tag{14}$$

Putting $u'(x) = k(x) s'(x)$, the above implies in particular that u' is integrable in a neighborhood of $+\infty$ such that $u(\infty)$ must be a finite number. We get $u'(x) = \gamma(x)(u(\infty) - u(x))$.

Case 1 : $u(\infty) = 0$ so that $u(x) = -c_1 e^{-\Gamma(x)}$ for some constant c_1, whence $\Gamma(\infty) = \infty$. We obtain

$$s'(x) = c_1 \frac{\gamma(x)}{k(x)} e^{-\Gamma(x)} \tag{15}$$

and thus

$$s\left(x\right)=c_{2}+c_{1}\int_{1}^{x}\frac{\gamma\left(y\right)}{k\left(y\right)}e^{-\Gamma\left(y\right)}dy \tag{16}$$

for some constants c_1, c_2. Taking $c_2 = 0$ and $c_1 = 1$ gives the formula (13), and both items (1.1) and (1.2) follow from this.

Case 2 : $u(\infty) \neq 0$ is a finite number. Putting $v(x) = e^{\Gamma(x)}u(x)$, v then solves

$$v'(x) = u(\infty)\gamma(x)e^{\Gamma(x)}$$

such that

$$v(x) = d_1 + u(\infty)e^{\Gamma(x)}$$

and thus

$$u(x) = e^{-\Gamma(x)}d_1 + u(\infty).$$

Letting $x \to \infty$, we see that the above is perfectly well-defined for any value of the constant d_1, if we suppose $\Gamma(\infty) = \infty$.

As a consequence, $u'(x) = -d_1\gamma(x)e^{-\Gamma(x)}$, leading us again to the explicit formula

$$s(x)=c_{2}+c_{1}\int_{1}^{x}\frac{\gamma\left(y\right)}{k\left(y\right)}e^{-\Gamma\left(y\right)}dy, \tag{17}$$

with $c_1 = -d_1$, implying items (1.1) and (1.2).

Finally, if $\Gamma(\infty) < \infty$, we see that we have to take $d_1 = 0$ implying that the only scale functions in this case are the constant ones. □

Example 5. In the linear case example with $\beta\left(x\right) = \beta_1 > 0$, $\alpha\left(x\right) = \alpha_1 x$, $\alpha_1 > 0$ and $k\left(y\right) = e^{-y}$, with $\gamma_1 = \beta_1/\alpha_1$, Assumption 2 is satisfied such that

$$s\left(x\right)=\gamma_{1}\int_{1}^{x}y^{-\left(\gamma_{1}+1\right)}e^{y}dy$$

which is diverging both as $x \to 0$ and as $x \to +\infty$. Notice that 0 is inaccessible for this process, i.e., starting from a strictly positive position $x > 0$, X_t will never hit 0. Defining the process on the state space $(0, \infty)$, invariant under the dynamics, the process is recurrent and even positive recurrent as we know from the Gamma shape of its invariant speed density.

3.1. Hitting times

Fix $a < x < b$. In what follows we shall be interested in hitting times of the positions a and b, starting from x, under the condition $\Gamma(\infty) = \infty$. Due to the asymmetric structure of the process (continuous motion downwards and up-moves by jumps only), these times are given by

$$\tau_{x,b} = \inf\{t > 0 : X_t = b\} = \inf\{t > 0 : X_t \le b, X_{t-} > b\}$$

and

$$\tau_{x,a} = \inf\{t > 0 : X_t = a\} = \inf\{t > 0 \ : \ X_t \le a\}.$$

Obviously, $\tau_{a,a} = \tau_{b,b} = 0$.

Let $T = \tau_{x,a} \wedge \tau_{x,b}$. Contrarily to the study of processes with continuous trajectories, it is not clear that $T < \infty$ almost surely. Indeed, starting from x, the process could jump across the barrier of height b before hitting a and then never enter the interval $[0, b]$ again. So we suppose in the sequel that $T < \infty$ almost surely. Then

$$\mathbb{P}\left(\tau_{x,a} < \tau_{x,b}\right) + \mathbb{P}\left(\tau_{x,b} < \tau_{x,a}\right) = 1.$$

Proposition 4. *Suppose* $\Gamma(\infty) = \infty$ *and that Assumption 2 does not hold. Let* $0 < a < x < b < \infty$ *and suppose that* $T = \tau_{x,a} \wedge \tau_{x,b} < \infty$ *almost surely. Then*

$$\mathbb{P}\left(\tau_{x,a} < \tau_{x,b}\right) = \frac{\int_x^b \frac{\gamma(y)}{k(y)} e^{-\Gamma(y)} dy}{\int_a^b \frac{\gamma(y)}{k(y)} e^{-\Gamma(y)} dy}. \tag{18}$$

Proof. Under the above assumptions, $s_1(X_t)$ is a local martingale and the stopped martingale $M_t = s_1(X_{T \wedge t})$ is bounded, which follows from the fact that $X_{T\wedge t} \ge a$ together with the observation that s_1 is decreasing implying that $M_t \le s_1(a)$. Therefore from the stopping theorem we have

$$s_1(x) = \mathbb{E}(M_0) = \mathbb{E}(s_1(X_0)) = \mathbb{E}(M_T).$$

Moreover,

$$\mathbb{E}(M_T) = s_1(a)\mathbb{P}(T = \tau_{x,a}) + s_1(b)\mathbb{P}(T = \tau_{x,b}).$$

But $\{T = \tau_{x,a}\} = \{\tau_{x,a} < \tau_{x,b}\}$ and $\{T = \tau_{x,b}\} = \{\tau_{x,b} < \tau_{x,a}\}$, such that

$$s_1(x) = s_1(a)\mathbb{P}(\tau_{x,a} < \tau_{x,b}) + s_1(b)\mathbb{P}(\tau_{x,b} < \tau_{x,a}).$$

□

Remark 8. - See [25] for similar arguments in a particular case of a constant flow and exponential jumps.

- We stress that it is not possible to deduce the above formula without imposing the existence of s_1 (that is, if Assumption 2 holds such that s_1 is not well-defined). Indeed, if we would want to consider the local martingale $s(X_t)$ instead, the stopped martingale $s(X_{t\wedge T})$ is not bounded since $X_{t\wedge T}$ might take arbitrary values in (a,∞), such that it is not possible to apply the stopping rule.

Remark 9. Let $\tau_{x,[b,\infty)} = \inf\{t>0 : X_t \geq b\}$ and $\tau_{x,[0,a]} = \inf\{t>0 : X_t \leq a\}$ be the entrance times to the intervals $[b,\infty)$ and $[0,a]$. Observe that by the structure of the process, namely the continuity of the downward motion,

$$\{\tau_{x,a} < \tau_{x,b}\} \subset \{\tau_{x,a} < \tau_{x,[b,\infty)}\} \subset \{\tau_{x,a} < \tau_{x,b}\}.$$

Indeed, the second inclusion is trivial since $\tau_{x,[b,\infty)} \leq \tau_{x,b}$. The first inclusion follows from the fact that it is not possible to jump across b and then hit a without touching b. Observing moreover that for $a<x$, $\tau_{x,a} = \tau_{x,[0,a]}$, (18) can therefore be rewritten as

$$\mathbb{P}\left(\tau_{x,[0,a]} < \tau_{x,[b,\infty)}\right) = \frac{s_1(x) - s_1(b)}{s_1(a) - s_1(b)}. \tag{19}$$

Now suppose that the process does not explode in finite time, that is, during each finite time interval, almost surely, only a finite number of jumps appear. In this case $\tau_{x,[b,\infty)} \to +\infty$ as $b\to\infty$. Then, letting $b\to\infty$ in (19), we obtain for any $a>0$,

$$\mathbb{P}\left(\tau_{x,a} < \infty\right) = \frac{s_1(x)}{s_1(a)} = \frac{\int_x^\infty \frac{\gamma(y)}{k(y)} e^{-\Gamma(y)} dy}{\int_a^\infty \frac{\gamma(y)}{k(y)} e^{-\Gamma(y)} dy} < 1, \tag{20}$$

since $\int_a^x \frac{\gamma(y)}{k(y)} e^{-\Gamma(y)} dy > 0$ by assumptions on k and γ.

As a consequence, we obtain the following

Proposition 5. *Suppose that $\Gamma(\infty)=\infty$ and that Assumption 2 does not hold. Then either the process explodes in finite time with positive probability or it is transient at $+\infty$, i.e. for all $a<x$, $\tau_{x,a}=\infty$ with positive probability.*

Proof. Let $a<x<b$ and $T = \tau_{x,a}\wedge\tau_{x,b}$ and suppose that almost surely, the process does not explode in finite time. We show that in this case, $\tau_{x,a}=\infty$ with positive probability.

Indeed, suppose that $\tau_{x,a} < \infty$ almost surely. Then (18) holds, and letting $b \to \infty$, we obtain (20) implying that $\tau_{x,a} = \infty$ with positive probability which is a contradiction. $\square$

Example 6. Choose $\alpha(x) = x^2$, $\beta(x) = 1 + x^2$ so that $\gamma(x) = 1 + 1/x^2$and $\Gamma(x) = x - 1/x$ with $\Gamma(0) = -\infty$ and $\Gamma(\infty) = \infty$.

Choose also $k(x) = e^{-x/2}$. Then Assumption 2 is violated: the survival function of the big upward jumps decays too slowly in comparison to the decay of $e^{-\Gamma}$. The speed density is

$$\pi(x) = C\frac{k(x)\,e^{\Gamma(x)}}{\alpha(x)} = Cx^{-2}e^{x/2-1/x}$$

which is integrable at 0 but not at ∞. The process explodes (has an infinite number of upward jumps in finite time) with positive probability.

3.2. First moments of hitting times

Let $a > 0$. We are looking for positive solutions of

$$(\mathcal{G}\phi_a)(x) = -1, x \geq a,$$

with boundary condition $\phi_a(a) = 0$. The above is equivalent to

$$(\mathcal{G}\phi_a)(x) = -\alpha(x)\,\phi_a'(x) + \frac{\beta(x)}{k(x)}\int_x^\infty k(y)\,\phi_a'(y)\,dy = -1.$$

This is also

$$-k(x)\,\phi_a'(x) + \gamma(x)\int_x^\infty k(y)\,\phi_a'(y)\,dy = -\frac{k(x)}{\alpha(x)}.$$

Putting $U(x) := \int_x^\infty k(y)\,\phi_a'(y)\,dy$, the latter integro-differential equation reads $U'(x) = -\gamma(x)\,U(x) - \frac{k(x)}{\alpha(x)}$. Supposing that $\int^{+\infty}\pi(y)dy < \infty$ (recall (12)), this leads to

$$\begin{aligned}
U(x) &= e^{-\Gamma(x)}\int_x^\infty e^{\Gamma(y)}\frac{k(y)}{\alpha(y)}dy, \\
-U'(x) &= \gamma(x)\,e^{-\Gamma(x)}\int_x^\infty e^{\Gamma(y)}\frac{k(y)}{\alpha(y)}dy + \frac{k(x)}{\alpha(x)} = k(x)\,\phi_a'(x),
\end{aligned}$$

such that

$$\begin{aligned}
\phi_a(x) &= \int_a^x dy\frac{\gamma(y)}{k(y)}e^{-\Gamma(y)}\int_y^\infty e^{\Gamma(z)}\frac{k(z)}{\alpha(z)}dz + \int_a^x \frac{dy}{\alpha(y)} \\
&= \int_a^\infty dz\pi(z)\left[s_1(a) - s_1(x\wedge z)\right] + \int_a^x \frac{dy}{\alpha(y)}. \qquad (21)
\end{aligned}$$

Notice that $[a,\infty) \ni x \mapsto \phi_a(x)$ is non-decreasing and that $\phi_a(x) < \infty$ for all $x > a > 0$ under our assumptions. Dynkin's formula implies that for all $x > a$ and all $t \geq 0$,

$$\mathbb{E}_x(t \wedge \tau_{x,a}) = \phi_a(x) - \mathbb{E}_x(\phi_a(X_{t\wedge\tau_{x,a}})). \tag{22}$$

In particular, since $\phi_a(\cdot) \geq 0$,

$$\mathbb{E}_x(t \wedge \tau_{x,a}) \leq \phi_a(x) < \infty,$$

such that we may let $t \to \infty$ in the above inequality to obtain by monotone convergence that

$$\mathbb{E}_x(\tau_{x,a}) \leq \phi_a(x) < \infty.$$

In a second step, we obtain from (22), using Fatou's lemma, that

$$\begin{aligned}\mathbb{E}_x(\tau_{x,a}) = \lim_{t\to\infty} \mathbb{E}_x(t \wedge \tau_{x,a}) = \phi_a(x) - \lim_{t\to\infty} \mathbb{E}_x(\phi_a(X_{t\wedge\tau_{x,a}}))\\ \geq \phi_a(x) - \mathbb{E}_x(\liminf_{t\to\infty} \phi_a(X_{t\wedge\tau_{x,a}})) = \phi_a(x),\end{aligned}$$

where we have used that $\liminf_{t\to\infty} \phi_a(X_{t\wedge\tau_{x,a}}) = \phi_a(a) = 0$.

As a consequence we have just shown the following

Proposition 6. *Suppose that* $\int^{+\infty} \pi(y)dy < \infty$. *Then* $\mathbb{E}_x(\tau_{x,a}) = \phi_a(x) < \infty$ *for all* $0 < a < x$, *where* ϕ_a *is given as in* (21).

Remark 10. The last term $\int_a^x \frac{dy}{\alpha(y)}$ in the RHS of the expression of $\phi_a(x)$ in (21) is the time needed for the deterministic flow to first hit a starting from $x > a$, which is a lower bound of $\phi_a(x)$. Considering the tail function of the speed density $\pi(y)$, namely $\overline{\pi}(y) := \int_y^\infty e^{\Gamma(z)} \frac{k(z)}{\alpha(z)} dz$, the first term in the RHS expression of $\phi_a(x)$ is

$$\int_a^x -ds_1(y)\,\overline{\pi}(y) = -\left[s_1(y)\,\overline{\pi}(y)\right]_a^x - \int_a^x s_1(y)\,\pi(y)\,dy,$$

emphasising the importance of the couple $(s_1(\cdot), \pi(\cdot))$ in the evaluation of $\phi_a(x)$. If a is a small critical value below which the population can be considered in danger, this is the mean value of a 'quasi-extinction' event when the initial size of the population was x.

Remark 11. Notice that the above discussion is only possible for couples $0 < a < x$, since starting from x, $X_{t\wedge\tau_{x,a}} \geq a$ for all t. A similar argument does not hold true for $x < b$ and the study of $\tau_{x,b}$.

3.3. Mean first hitting time of 0

Suppose that $\Gamma(0) > -\infty$. Then for flows $x_t(x)$ that go extinct in finite time $t_0(x)$, under the condition that $\int^{+\infty} \pi(y)dy < \infty$, one can let $a \to 0$ in the expression of $\phi_a(x)$ to obtain

$$\phi_0(x) = \int_0^x dy \frac{\gamma(y)}{k(y)} e^{-\Gamma(y)} \int_y^\infty e^{\Gamma(z)} \frac{k(z)}{\alpha(z)} dz + \int_0^x \frac{dy}{\alpha(y)},$$

which is the expected time to eventual extinction of X starting from x, that is, $\phi_0(x) = \mathbb{E}\tau_{x,0}$.

The last term $\int_0^x \frac{dy}{\alpha(y)} = t_0(x) < \infty$ in the RHS expression of $\phi_0(x)$ is the time needed for the deterministic flow to first hit 0 starting from $x > 0$, which is a lower bound of $\phi_0(x)$.

Notice that under the condition $t_0(x) < \infty$ and $k(0) < \infty$, $\int_0 \pi(y)dy < \infty$, such that π can be tuned into a probability. It is easy to see that $\Gamma(0) > -\infty$ implies then that $\phi_0(x) < \infty$.

Example 7. (Linear release at constant jump rate): Suppose $\alpha(x) = \alpha_1 > 0$, $\beta(x) = \beta_1 > 0$, $\gamma(x) = \gamma_1 = \beta_1/\alpha_1$, $\Gamma(x) = \gamma_1 x$ with $\Gamma(0) = 0 > -\infty$. Choose $k(x) = e^{-x}$.

State 0 is reached in finite time $t_0(x) = x/\alpha_1$, and it turns out to be reflecting. We have $\int_0 \pi(x)\,dx < \infty$ and $\int^\infty \pi(x)\,dx < \infty$ if and only if $\gamma_1 < 1$. In such a case, the first integral term in the above expression of $\phi_0(x)$ is $\gamma_1/[\alpha_1(1-\gamma_1)]\,x$, so that $\phi_0(x) = x/[\alpha_1(1-\gamma_1)] < \infty$.

4. Non-explosion and Recurrence

In this section we come back to the scale function s introduced in (13) above. Despite the fact that we cannot use s to obtain explicit expressions for exit probabilities, we show how we might use it to obtain Foster-Lyapunov criteria in spirit of Meyn and Tweedie (1993) that imply the non-explosion of the process together with its recurrence under additional irreducibility properties.

Let $S_1 < S_2 < \ldots < S_n < \ldots$ be the successive jump times of the process and $S_\infty = \lim_{n\to\infty} S_n$. We start discussing how we can use the scale function s to obtain a general criterion for non-explosion of the process, that is, $S_\infty = +\infty$ almost surely.

Proposition 7. *Suppose $\Gamma(\infty) = \infty$ and suppose that Assumption 2 holds. Suppose also that β is continuous on $[0, \infty)$. Let V be any $\mathcal{C}^1-$function defined on $[0, \infty)$, such that $V(x) = 1 + s(x)$ on $[1, \infty)$ and such that $V(x) \geq 1/2$ for all x. Then V is a norm-like function in the sense of Meyn and Tweedie (1993), and we have*

1. $\mathcal{G}V(x) = 0$, $\forall x \geq 1$.

2. $\sup_{x \in [0,1]} |\mathcal{G}V(x)| < \infty$.

As a consequence, $S_\infty = \sup_n S_n = \infty$ almost surely, so that X is non-explosive.

Proof. We check that V satisfies the condition (CD0) of [27]. It is evident that V is norm-like since $\lim_{x \to \infty} V(x) = 1 + \lim_{x \to \infty} s(x) = 1 + s(\infty) = \infty$ since Assumption 2 holds. Moreover, since $K(x, y) = \frac{k(y)}{k(x)}$,

$$\mathcal{G}V(x) = -\alpha(x)V'(x) + \frac{\beta(x)}{k(x)} \int_x^\infty k(y)V'(y)dy.$$

Since for all $1 \leq x \leq y$, $V'(x) = s'(x)$ and $V'(y) = s'(y)$, we have $\mathcal{G}V = \mathcal{G}s = 0$ on $[1, \infty[$.

For the second point, for $x \in]0, 1[$,

$$\begin{aligned} \mathcal{G}V(x) &= -\alpha(x)V'(x) + \frac{\beta(x)}{k(x)} \int_x^\infty k(y)V'(y)dy \\ &= -\alpha(x)V'(x) + \frac{\beta(x)}{k(x)} \int_1^\infty k(y)V'(y)dy + \beta(x) \int_x^1 K(x, y)V'(y)dy. \end{aligned}$$

$\alpha(x)V'(x)$ is continuous and thus bounded on $[0, 1]$. Moreover, for all $y \geq x$, $K(x, y) \leq 1$ implying that $\beta(x) \int_x^1 K(x, y)V'(y)dy \leq \beta(x) \int_x^1 V'(y)dy < \infty$. We also have for all $y \geq 1$,

$$\int_1^\infty k(y)V'(y)dy = \int_1^\infty k(y)s'(y)dy.$$

Thus, using $\mathcal{G}V(x) = \mathcal{G}s(x) = 0$ on $]0, 1[$ we have

$$\frac{\beta(x)}{k(x)} \int_1^\infty k(y)V'(y)dy = \frac{\beta(x)}{k(x)} k(1)V'(1)/\gamma(1) < \infty$$

because β is continuous on $[0, 1]$ and k taking finite values on $(0, \infty)$. As a consequence, $\sup_{x \in [0,1]} |\mathcal{G}V(x)| < \infty$. $\square$

We close this subsection with a stronger Foster-Lyapunov criterion implying the existence of finite hitting time moments.

Proposition 8. *Suppose there exist $x_* > 0$, $c > 0$ and a positive function V such that $\mathcal{G}V(x) \le -c$ for all $x \ge x_*$. Then for all $x \ge a \ge x_*$,*

$$\mathbb{E}(\tau_{x,a}) \le \frac{V(x)}{c}.$$

Proof. Using Dynkin's formula, we have for $x \ge a \ge x_*$,

$$\mathbb{E}(V(X_{t\wedge\tau_{x,a}})) = V(x) + \mathbb{E}(\int_0^{t\wedge\tau_{x,a}} \mathcal{G}V(X_s)ds) \le V(x) - c\mathbb{E}(t\wedge\tau_{x,a}),$$

such that

$$\mathbb{E}(t\wedge\tau_{x,a}) \le \frac{V(x)}{c},$$

which implies the assertion, letting $t \to \infty$. □

Example 8. Suppose $\alpha(x) = 1 + x$, $\beta(x) = x$ and $k(x) = e^{-2x}$. We choose $V(x) = e^x$. Then

$$(\mathcal{G}V)(x) = -e^x \le -1, \forall x \ge 0.$$

As a consequence $\mathbb{E}(\tau_{x,a}) < \infty$, for all $a > 0$.

4.1. Irreducibility and Harris recurrence

In this section we impose that $\Gamma(\infty) = \infty$, such that non-trivial scale functions do exist. We also assume that Assumption 2 holds since otherwise the process is either transient at ∞ or explodes in finite time. Then the function V introduced in Proposition 7 is a Lyapunov function. This is *almost* the Harris recurrence of the process, all we need to show is some irreducibility property that we are going to check now.

Theorem 1. *Suppose we are in the separable case, that $k \in C^1$ and that 0 is inaccessible, that is, $t_0(x) = \infty$ for all x. Then every compact set $C \subset]0,\infty[$ is 'petite' in the sense of Meyn and Tweedie. More precisely, there exist $t > 0$, $\alpha \in (0,1)$ and a probability measure ν on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, such that*

$$P_t(x,dy) \ge \alpha \mathbf{1}_C(x)\nu(dy).$$

Proof. Suppose w.l.o.g. that $C = [a,b]$ with $0 < a < b$. Fix any $t > 0$. The idea of our construction is to impose that all processes $X_s(x), a \le x \le b$, have one single common jump during $[0,t]$. Indeed, notice that for each $x \in C$, the jump rate of $X_s(x)$ is given by $\beta(x_s(x))$ taking values in a compact set $[\beta_*, \beta^*]$ where $\beta_* = \min\{\beta(x_s(x)), 0 \le s \le t, x \in C\}$ and $\beta^* = \max\{\beta(x_s(x)), 0 \le s \le t, x \in C\}$. Notice that $0 < \beta_* < \beta^* < \infty$ since β is supposed to be positive on $(0, \infty)$. We then construct all processes $X_s(x), s \le t, x \in C$, using the same underlying Poisson random measure M. It thus suffices to impose that E holds, where

$$E = \{M([0, t-\varepsilon] \times [0, \beta^*]) = 0, M([t-\varepsilon, t] \times [0, \beta_*]) = 1, M([t-\varepsilon, t] \times]\beta_*, \beta^*]) = 0\}.$$

Indeed, the above implies that up to time $t - \varepsilon$, none of the processes $X_s(x), x \in C$, jumps. The second and third assumption imply moreover that the unique jump time, call it S, of M within $[t-\varepsilon, t] \times [0, \beta^*]$ is a common jump of all processes. For each value of $x \in C$, the associated process $X(x)$ then chooses a new after-jump position y according to

$$\frac{1}{k(x_S(x))} |k'(y)| dy \mathbf{1}_{\{y \ge x_S(x)\}}. \tag{23}$$

Case 1. Suppose k is strictly decreasing, that is, $|k'|(y) > 0$ for all y. Fix then any open ball $B \subset [b, \infty[$ and notice that $\mathbf{1}_{\{y \ge x_S(x)\}} \ge \mathbf{1}_B(y)$, since $b \ge x_S(x)$. Moreover, since k is decreasing, $1/k(x_S(x)) \ge 1/k(x_t(a))$. Therefore, the transition density given in (23) can be lower-bounded, independently of x, by

$$\frac{1}{k(x_t(a))} \mathbf{1}_B(y) |k'(y)| dy = p \tilde{\nu}(dy),$$

where $\tilde{\nu}(dy) = c \mathbf{1}_B(y) |k'(y)| dy$, normalised to be a probability density, and where $p = \frac{1}{k(x_t(a))}/c$. In other words, on the event E, with probability p, all particles choose a new and common position $y \sim \tilde{\nu}(dy)$ and couple.

Case 2. $|k'|$ is different from 0 on a ball B (but not necessarily on the whole state space). We suppose w.l.o.g. that B has compact closure. Then it suffices to take t sufficiently large in the first step such that $x_{t-\varepsilon}(b) < \inf B$. Indeed, this implies once more that $\mathbf{1}_B(y) \le \mathbf{1}_{\{y \ge x_s(x)\}}$ for all $x \in C$ and for s the unique common jump time.

Conclusion. In any of the above cases, let $\bar{b} := \sup\{x : x \in B\} < \infty$ and restrict the set E to

$$E' = E \cap \{M([t-\varepsilon] \times]\beta_*, \bar{b}] = 0\}.$$

Putting $\alpha := p \Pr(E')$ and

$$\nu(dy) = \int_{t_\varepsilon}^t \mathcal{L}(S|E')(ds) \int \tilde{\nu}(dz) \delta_{x_{t-s}(z)}(dy)$$

then allows to conclude. □

Remark 12. If β is continuous on $[0,\infty)$ with $\beta(0)>0$ and if moreover $k(0)<\infty$, the above construction can be extended to any compact set of the form $[0,b], b<\infty$ and to the case where $t_0(x)<\infty$.

As a consequence of the above considerations we obtain the following theorem.

Theorem 2. *Suppose that $\beta(0)>0, k(0)<\infty$, that $\Gamma(\infty)=\infty$ and moreover that Assumption 2 holds. Then the process is recurrent in the sense of Harris and its unique invariant measure is given by π.*

Proof. Condition (CD1) of Meyn and Tweedie (1993) holds with V given as in Proposition 7 and with compact set $C=[0,1]$. By Theorem 1, all compact sets are 'petite'. Then Theorem 3.2 of [27] allows to conclude. □

Example 9. A meaningful recurrent example consists of choosing $\beta(x)=\beta_1/x$, $\beta_1>0$ (the surge rate decreases like $1/x$), $\alpha(x)=\alpha_1/x$ (finite time extinction of x_t), $\alpha_1>0$, and $k(y)=e^{-y}$. In this case, all compact sets are 'petite'. Moreover, with $\gamma_1=\beta_1/\alpha_1$, $\Gamma(x)=\gamma_1 x$ and

$$\pi(y)=\frac{y}{\alpha_1}e^{(\gamma_1-1)y}$$

which can be tuned into a probability density if $\gamma_1<1$. This also implies that Assumption 2 is satisfied such that s can be used to define a Lyapunov function. The associated process X is positive recurrent if $\gamma_1<1$, null-recurrent if $\gamma_1=1$.

5. The embedded chain

In this section, we illustrate some of the previously established theoretical results by simulations of the embedded chain that we are going to define now. Defining $T_1=S_1, T_n=S_n-S_{n-1}, n\geq 2$, the successive inter-jump waiting times, we have

$$\begin{aligned}\mathbb{P}\left(T_n\in dt, X_{S_n}\in dy \mid X_{S_{n-1}}=x\right) &= dt\beta\left(x_t\left(x\right)\right)e^{-\int_0^t\beta(x_s(x))ds}K\left(x_t\left(x\right),dy\right)\\ &= dt\beta\left(x_t\left(x\right)\right)e^{-\int_{x_t(x)}^x\gamma(z)dz}K\left(x_t\left(x\right),dy\right).\end{aligned}$$

The embedded chain is then defined through $Z_n := X_{S_n}, n \geq 0$. If 0 is not absorbing, for all $x \geq 0$

$$\begin{aligned} \mathbb{P}\left(Z_n \in dy \mid Z_{n-1} = x\right) &= \int_0^\infty dt\beta\left(x_t\left(x\right)\right) e^{-\int_{x_t(x)}^{x} \gamma(z)dz} K\left(x_t\left(x\right), dy\right) \\ &= e^{-\Gamma(x)} \int_0^x dz\gamma\left(z\right) e^{\Gamma(z)} K\left(z, dy\right), \end{aligned}$$

where the last line is valid for $x > 0$ only, and only if $t_0(x) = \infty$. This implies that Z_n is a time-homogeneous discrete-time Markov chain on $[0, \infty)$.

Remark 13. $\left(S_n, Z_n\right)_{n \geq 0}$ is also a discrete-time Markov chain on $\mathbb{R}_+^2$ with transition probabilities given by

$$\mathbb{P}\left(S_n \in dt \mid Z_{n-1} = x, S_{n-1} = s\right) = dt\beta\left(x_{t-s}\left(x\right)\right) e^{-\int_0^{t-s} \beta(x_{s'}(x))ds'}, \; t \geq s,$$

and

$$\mathbb{P}\left(Z_n \in dy \mid Z_{n-1} = x, S_{n-1} = s\right) = \int_0^\infty dt\beta\left(x_t\left(x\right)\right) e^{-\int_{x_t(x)}^{x} \gamma(z)dz} K\left(x_t\left(x\right), dy\right)$$

independent of s. Note

$$\mathbb{P}\left(T_n \in d\tau \mid Z_{n-1} = x\right) = d\tau\beta\left(x_\tau\left(x\right)\right) e^{-\int_0^\tau \beta(x_s(x))ds}, \tau \geq 0.$$

Coming back to the marginal Z_n and assuming $\Gamma\left(0\right) = -\infty$, the arguments of Sections 2.5 and 2.6 in [15] imply that

$$\begin{aligned} \mathbb{P}\left(Z_n > y \mid Z_{n-1} = x\right) &= e^{-\Gamma(x)} \int_0^x dz\gamma\left(z\right) e^{\Gamma(z)} \int_y^\infty K\left(z, dy'\right) \\ &= 1 - e^{\Gamma(x\wedge y) - \Gamma(x)} + e^{-\Gamma(x)} \int_0^{x\wedge y} dz\gamma\left(z\right) e^{\Gamma(z)} K\left(z, y\right). \end{aligned} \tag{24}$$

To obtain the last line, we have used that $K\left(z, y\right) = 1$ for all $y \leq z$ and, whenever $z < y$, we have split the second integral in the first line into the two parts corresponding to ($z < y \leq x$ and $z \leq x < y$).

To simulate the embedded chain, we have to decide first if, given $Z_{n-1} = x$, the forthcoming move is down or up.

- A move up occurs with probability given by

$$\mathbb{P}\left(Z_n > x \mid Z_{n-1} = x\right) = e^{-\Gamma(x)} \int_0^x dz\gamma\left(z\right) e^{\Gamma(z)} K\left(z, x\right).$$

- A move down occurs with complementary probability.

As soon as the type of move is fixed (down or up), to decide where the process goes precisely, we must use the inverse of the corresponding distribution function (24) (with $y \le x$ or $y > x$), conditioned on the type of move.

Remark 14. If state 0 is absorbing, Eq. (24) is valid only when $x > 0$, and the boundary condition $\mathbb{P}\left(Z_n = 0 \mid Z_{n-1} = 0\right) = 1$ should be added.

In the following simulations, as before, we work in the separable case $K(x,y) = \frac{k(y)}{k(x)}$, where we choose $k(x) = e^{-x}$ in the first simulation and $k(x) = 1/(1+x^2)$ in the second. Moreover, we take $\alpha(x) = \alpha_1 x^a$ and $\beta(x) = \beta_1 x^b$ with $\alpha_1 = 1$, $a = 2$, $\beta_1 = 1$ and $b = 1$. In these cases, there is no finite time extinction of the process $x_t(x)$; that is, in both cases, state 0 is not accessible.

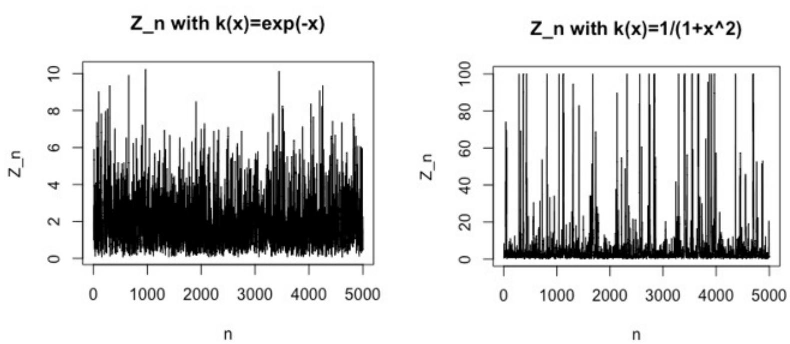


FIGURE 1: Notice that in accordance with the fact that $k(x) = 1/(1+x^2)$ has slower decaying tails than $k(x) = e^{-x}$, the process with jump distribution $k(x) = 1/(1+x^2)$ has higher maxima than the process with $k(x) = e^{-x}$.

The graphs above do not provide any information about the jump times. In what follows we take this additional information into account and simulate the values Z_n of the embedded process as a function of the jump times S_n. To do so we must calculate the distribution $\mathbb{P}\left(S_n \le t \mid X_{S_{n-1}} = x, S_{n-1} = s\right)$. Using Remark 13 we have

$$\mathbb{P}\left(S_n \le t \mid X_{S_{n-1}} = x, S_{n-1} = s\right) = \int_s^t dt' \beta\left(x_{t'-s}(x)\right) e^{-\int_0^{t'-s} \beta(x_{s'}(x))ds'}$$

$$= \int_0^{t-s} du \beta\left(x_u(x)\right) e^{-\int_0^u \beta(x_{s'}(x))ds'} = 1 - e^{-\int_0^{t-s} \beta(x_{s'}(x))ds'} = 1 - e^{-\left[\Gamma(x) - \Gamma(x_{t-s}(x))\right]}.$$

The simulation of the jump times S_n then goes through a simple inversion of the conditional distribution function $\mathbb{P}\left(S_n \leq t \mid X_{S_{n-1}} = x, S_{n-1} = s\right)$. In the following simulations we use the same parameters as in the previous simulations.

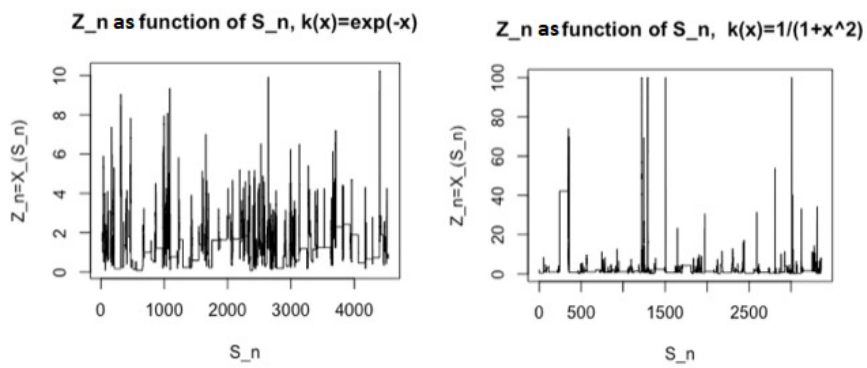


FIGURE 2: The above graphs give the positions Z_n as a function of the jump times S_n. The waiting times between successive jumps are longer in the first process than in the second one. Since we use the same jump rate function in both processes and since this rate is an increasing function of the positions, this is due to the fact that jumps lead to higher values in the second process than in the first such that jumps occur more frequently.

6. The extremal record chain

Of interest are the upper record times and values sequences of Z_n, namely

$$\begin{aligned} R_n &= \inf\left(r \geq 1 : r > R_{n-1}, Z_r > Z_{R_{n-1}}\right) \\ Z_n^* &= Z_{R_n}. \end{aligned}$$

Unless X (and so Z_n) goes extinct, Z_n^* is a strictly increasing sequence tending to ∞.

Following [1], with $(R_0 = 0, Z_0^* = x)$, $(R_n, Z_n^*)_{n\geq 0}$ clearly is a Markov chain with transition probabilities for $y > x$

$$\begin{aligned} \overline{P}^*(k, x, y) &:= \mathbb{P}\left(R_n = r + k, Z_n^* > y \mid R_{n-1} = r, Z_{n-1}^* = x\right) \\ &= \overline{P}(x, y) \text{ if } k = 1 \\ &= \int_0^x \dots \int_0^x \prod_{l=0}^{k-2} P(x_l, dx_{l+1}) \, \overline{P}(x_{k-1}, y) \text{ if } k \geq 2, \end{aligned}$$

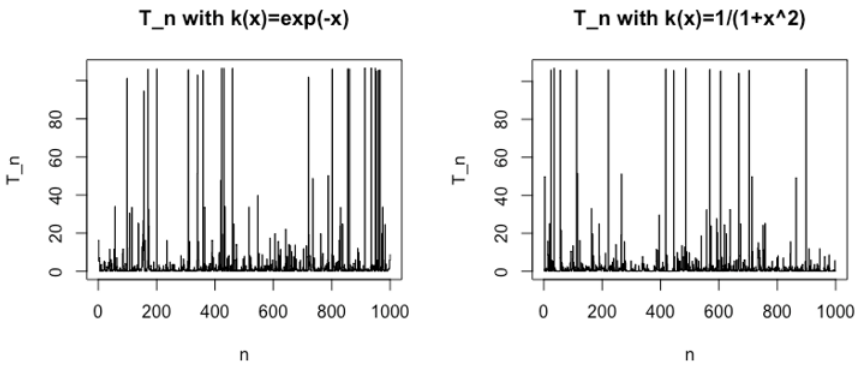


FIGURE 3: These graphs represent the sequence $T_n = S_n - S_{n-1}$ of the inter jump waiting times for the two processes showing once more that these waiting times are indeed longer in the first process than in the second.

where $P(x, dy) = \mathbb{P}(Z_n \in dy \mid Z_{n-1} = x)$, $\overline{P}(x, y) = \mathbb{P}(Z_n > y \mid Z_{n-1} = x)$ and $x_0 = x$.

Clearly the marginal sequence $(Z_n^*)_{n\geq 0}$ is Markov with transition matrix

$$\overline{P}^*(x, y) := \mathbb{P}\left(Z_n^* > y \mid Z_{n-1}^* = x\right) = \sum_{k\geq 1} \overline{P}^*(k, x, y),$$

but the record times marginal sequence $(R_n)_{n\geq 0}$ is non-Markov. However

$$\mathbb{P}\left(R_n = r + k \mid R_{n-1} = r, Z_{n-1}^* = x\right) = \overline{P}^*(k, x, x),$$

showing that the law of $A_n := R_n - R_{n-1}$ (the age of the n-th record) is independent of R_{n-1} (although not of Z_{n-1}^*):

$$\mathbb{P}\left(A_n = k \mid Z_{n-1}^* = x\right) = \overline{P}^*(k, x, x),\ k \geq 1.$$

Of particular interest is $(R_1, Z_1^* = Z_{R_1})$, the first upper record time and value because S_{R_1} is the first time $(X_t)_t$ exceeds the threshold x, and Z_{R_1} the corresponding overshoot at $y > x$. Its joint distribution is simply $(y > x)$

$$P^*(k, x, dy) = \mathbb{P}(R_1 = k, Z_1^* \in dy \mid R_0 = 0, Z_0^* = x)$$

$$
\begin{aligned}
&= P\left(x,dy\right) \text{ if } k=1 \\
&= \int_0^x ... \int_0^x \prod_{l=0}^{k-2} P\left(x_l, dx_{l+1}\right) P\left(x_{k-1}, dy\right) \text{ if } k \geq 2.
\end{aligned}
$$

If $y_c > x$ is a critical threshold above which one wishes to evaluate the joint probability of $(R_1, Z_1^* = Z_{R_1})$, then $P^*\left(k,x,dy\right)/\overline{P}^*\left(k,x,y_c\right)$ for $y > y_c$ is its right expression.

Note that $\mathbb{P}\left(R_1 = k \mid R_0 = 0, Z_0^* = x\right) = \overline{P}^*\left(k,x,x\right)$ and also that

$$P^*\left(x,dy\right) := \mathbb{P}\left(Z_1^* \in dy \mid Z_0^* = x\right) = \sum_{k \geq 1} P^*\left(k,x,dy\right).$$

Of interest is also the number of records in the set $\{0,...,N\}$:

$$\mathcal{R}_N := \#\left\{n \geq 0 : R_n \leq N\right\} = \sum_{n \geq 0} \mathbf{1}_{\{R_n \leq N\}}.$$

FIGURE 4: The following graphs represent the records of the two processes as a function of the ranks of the records, the one at the left with $k(x) = e^{-x}$ and the one at the right with $k(x) = 1/(1+x^2)$. We can notice that there are more records in the first graph than in the second. Records occur more frequently in the first graph than in the second. On the other hand, the heights of the records are much lower in the first graph than in the second.

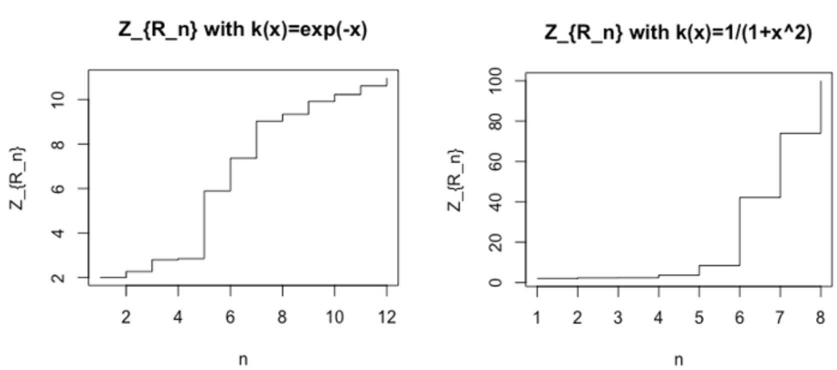


FIGURE 5: The following graphs give Z_{R_n} as a function of R_n. We remark that the gap between two consecutive records decreases over the time whereas the time between two consecutive records becomes longer. In other words, the higher is a record, the longer it takes to surpass it statistically.

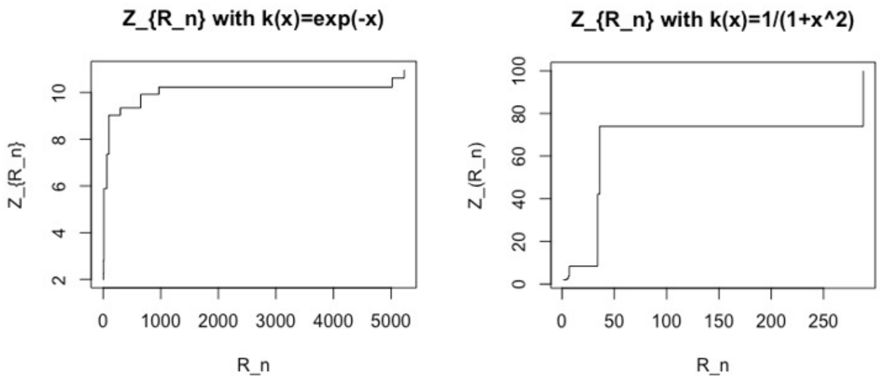


FIGURE 6: The following graphs give $A_n = R_n - R_{n-1}$ as a function of n. The differences between two consecutive records are much greater in the first graph than in the second. It is also noted that the maximum time gap is reached between the penultimate record and the last record.

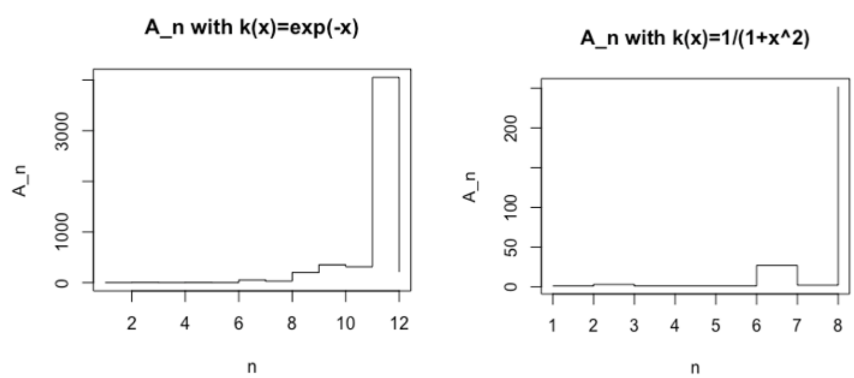
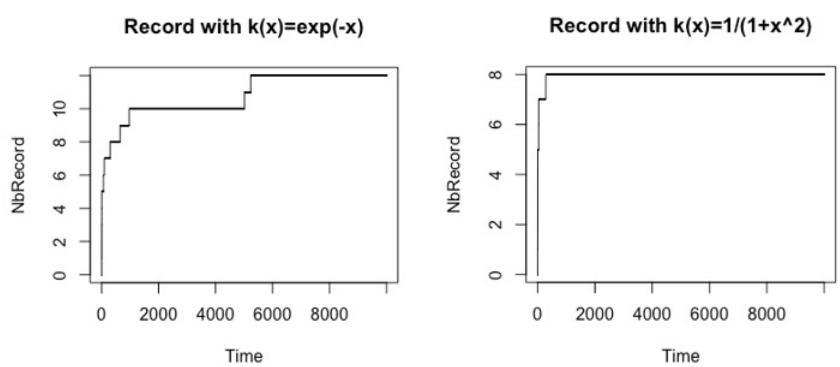


Figure 7: The following graphs give the obtained records from the simulation of the two processes, as a function of time. We can remark that the curve is slowly increasing with the time. In fact, to reach the 12th record, the first simulated process has needed 5500 units of time and analogously, the second simulated process has needed 1500 units of time to reach its 8th record.



7. Decay-Surge processes and the relation with Hawkes and Shot-noise processes

7.1. Hawkes processes

In the section we study the particular case $\beta(x) = \beta_1 x$, $\beta_1 > 0$ (the surge rate increases linearly with x), $\alpha(x) = \alpha_1 x$, $\alpha_1 > 0$ (exponentially declining population) and $k(y) = e^{-y}$. In this case, with $\gamma_1 = \beta_1/\alpha_1$, $\Gamma(x) = \gamma_1 x$.

In this case, we remark $\Gamma(0) = 0 > -\infty$. Therefore there is a strictly positive probability that the process will never jump (in which case it is attracted to 0). However we have $t_0(x) = \infty$, so the process never hits 0 in finite time. Finally, $\beta(0) = 0$ implies that state 0 is natural (absorbing and inaccessible).

Note that for this model,

$$\pi(y) = \frac{1}{\alpha_1 y} e^{(\gamma_1 - 1)y}$$

and we may take a version of the scale function given by

$$s(x) = \frac{\gamma_1}{1-\gamma_1} [e^{-(\gamma_1 - 1)y}]_0^x = \frac{\gamma_1}{1-\gamma_1} \left(e^{(1-\gamma_1)x} - 1\right).$$

Clearly, Assumption 2 is satisfied if and only if $\gamma_1 < 1$. We call the case $\gamma_1 < 1$ *subcritical*, the case $\gamma_1 > 1$ *supercritical* and the case $\gamma_1 = 1$ *critical*.

Supercritical case. It can be shown that the process does not explode almost surely, such that it is transient in this case (see Proposition 5). The speed density is neither integrable at 0 nor at ∞.

Critical and subcritical case. If $\gamma_1 < 1$, then π is integrable at $+\infty$, and we find

$$\phi_a(x) = -\left[s(y)\overline{\pi}(y)\right]_a^x - \int_a^x s(y)\pi(y)\,dy + \frac{1}{\alpha_1}\log\frac{x}{a},$$

where (with Ei the exponential integral function)

$$\int_a^x s(y)\pi(y)\,dy = \frac{\gamma_1}{\alpha_1(1-\gamma_1)}\left[\ln\left(\frac{x}{a}\right) - \mathrm{Ei}\left((\gamma_1-1)x\right) + \mathrm{Ei}\left((\gamma_1-1)a\right)\right].$$

In the critical case $\gamma_1 = 1$ the hitting time of a will be finite without having finite expectation.

In both critical and subcritical cases, that is, when $\gamma_1 \le 1$, the process X_t converges to 0 as $t \to \infty$ as we shall show now.

Due to the additive structure of the underlying deterministic flow and the exponential jump kernel, we have the explicit representation

$$X_t = e^{-\alpha_1 t}x + \sum_{n\ge 1: S_n \le t} e^{-\alpha_1(t-S_n)}Y_n, \tag{25}$$

where the $(Y_n)_{n\ge1}$ are i.i.d. exponentially distributed random variables with mean 1, such that for all n, Y_n is independent of $S_k, k \le n$, and of $Y_k, k < n$. Finally, in (25), the process X_t jumps at rate $\beta_1 X_{t-}$.

The above system is a *linear Hawkes process* without immigration, with kernel function $h(t) = e^{-\alpha_1 t}$ and with random jump heights $(Y_n)_{n\ge1}$ (see [22], see also [5]). Such a Hawkes process can be interpreted as *inhomogeneous Poisson process with branching*. Indeed, the additive structure in (25) suggests the following construction.

- At time 0, we start with a Poisson process having time-dependent rate $\beta_1 e^{-\alpha_1 t}x$.
- At each jump time S of this process, a new (time inhomogeneous) Poisson process is born and added to the existing one. This new process has intensity $\beta_1 e^{-\alpha_1(t-S)}Y$, where Y is exponentially distributed with parameter 1, independent of what has happened before. We call the jumps of this newborn Poisson process *jumps of generation* 1.

- At each jump time of generation 1, another time inhomogeneous Poisson process is born, of the same type, independently of anything else that has happened before. This gives rise to jumps of generation 2.
- The above procedure is iterated until it eventually stops since the remaining Poisson processes do not jump any more.

The total number of jumps of any of the offspring Poisson processes is given by

$$\beta_1 \mathbb{E}(Y) \int_S^\infty e^{-\alpha_1(t-S)} dt = \gamma_1.$$

So we see that whenever $\gamma_1 \leq 1$, we are considering a subcritical or critical Galton-Watson process which goes extinct almost surely, after a finite number of reproduction events. This extinction event is equivalent to the fact that the total number of jumps in the system is finite almost surely, such that after the last jump, X_t just converges to 0 (without, however, ever reaching it). Notice that in the subcritical case $\gamma_1 < 1$, the speed density is integrable at ∞, while it is not at 0 corresponding to absorption in 0.

An interesting feature of this model is that it can exhibit a phase transition when γ_1 crosses the value 1.

Finally, in the case of a linear Hawkes process with immigration we have $\beta(x) = \mu + \beta_1 x$, with $\mu > 0$. In this case,

$$\pi(y) = \frac{1}{\alpha_1} e^{(\gamma_1 - 1)y} y^{\mu/\alpha_1 - 1}$$

which is always integrable in 0 and which can be tuned into a probability in the subcritical case $\gamma_1 < 1$ corresponding to positive recurrence.

Remark 15. An interpretation of the decay-surge process in terms of Hawkes processes is only possible in case of affine jump rate functions β, additive drift α and exponential kernels k as considered above.

7.2. Shot-noise processes

Let $h(t)$, $t \geq 0$, with $h(0) = 1$ be a causal non-negative non-increasing response function translating the way shocks will attenuate as time passes by in a shot-noise process. We assume $h(t) \to 0$ as $t \to \infty$ and

$$\int_0^\infty h(s)\, ds < \infty. \tag{26}$$

With $X_0 = x \geq 0$, consider then the linear shot-noise process

$$X_t = x + \int_0^t \int_{\mathbb{R}_+} yh\left(t-s\right)\mu\left(ds, dy\right), \tag{27}$$

where, with $(S_n; n \geq 1)$ the points of a homogeneous Poisson point process with intensity β, $\mu\left(ds, dy\right) = \sum_{n\geq 1} \delta_{S_n}\left(ds\right)\delta_{\Delta_n}\left(dy\right)$ (translating independence of the shots' heights Δ_n and occurrence times S_n). Note that, with $dN_s = \sum_{n\geq 1} \Delta_n \delta_{S_n}\left(ds\right)$, so with $N_t = \sum_{n\geq 1} \Delta_n \mathbf{1}_{\{S_n \leq t\}}$ representing a time-homogeneous compound Poisson process with jumps' amplitudes Δ,

$$X_t = x + \int_0^t h\left(t-s\right) dN_s \tag{28}$$

is a linearly filtered compound Poisson process. Under this form, it is clear that X_t cannot be Markov unless $h\left(t\right) = e^{-\alpha t}$, $\alpha > 0$. We define

$$\begin{aligned} \nu\left(dt, dy\right) &= \mathbb{P}\left(S_n \in dt, \Delta_n \in dy \text{ for some } n \geq 1\right) \\ &= \beta dt \cdot \mathbb{P}\left(\Delta \in dy\right). \end{aligned}$$

In the sequel, we shall assume without much loss of generality that $x = 0$.

The linear shot-noise process X_t has two alternative equivalent representations, emphasising its **superposition** characteristics:

$$\begin{aligned} (1)\ \ X_t &= \sum_{n\geq 1} \Delta_n h\left(t - S_n\right)\mathbf{1}_{\{S_n \leq t\}} \\ (2)\ \ X_t &= \sum_{p=1}^{P_t} \Delta_p h\left(t - \mathcal{S}_p\left(t\right)\right), \end{aligned}$$

where $P_t = \sum_{n\geq 1} \mathbf{1}_{\{S_n \leq t\}}$.

Both show that X_t is the size at t of the whole decay-surge population, summing up all the declining contributions of the sub-families which appeared in the past at jump times (a shot-noise or filtered Poisson process model appearing also in Physics and Queuing theory, [32], [28]). The contributions $\Delta_p h\left(t - \mathcal{S}_p\left(t\right)\right)$, $p = 1, ...P_t$, of the P_t families to X_t are stochastically ordered in decreasing sizes.

In the Markov case, $h\left(t\right) = e^{-\alpha t}$, $t \geq 0$, $\alpha > 0$, we have

$$X_t = e^{-\alpha t} \int_0^t e^{\alpha s} dN_s, \tag{29}$$

so that

$$dX_t = -\alpha X_t dt + dN_t,$$

showing that X_t is a time-homogeneous Markov process driven by N_t, known as the **classical** linear shot-noise. This is clearly the only choice of the response function that makes X_t Markov. In that case, by Campbell's formula (see [28]),

$$\begin{aligned} \Phi_t^X(q) \quad &: \quad = \mathbb{E}e^{-qX_t} = e^{-\beta \int_0^t \left(1-\phi_\Delta\left(qe^{-\alpha(t-s)}\right)\right) ds} \\ &= \quad e^{-\frac{\beta}{\alpha}\int_{e^{-\alpha t}}^1 \frac{1-\phi_\Delta(qu)}{u} du} \text{ where } e^{-\alpha s} = u, \end{aligned}$$

with

$$\Phi_t^X(q) \to \Phi_\infty^X(q) = e^{-\frac{\beta}{\alpha}\int_0^1 \frac{1-\phi_\Delta(qu)}{u} du} \text{ as } t \to \infty.$$

The simplest explicit case is when $\phi_\Delta(q) = 1/(1+q/\theta)$ (else $\Delta \sim$Exp(θ)) so that, with $\gamma = \beta/\alpha$,

$$\Phi_\infty^X(q) = (1+q/\theta)^{-\gamma}$$

the Laplace-Stieltjes-Transform of a Gamma(γ,θ) distributed random variable X_∞, with density

$$\frac{\theta^\gamma}{\Gamma(\gamma)} x^{\gamma-1} e^{-\theta x},\ x > 0.$$

This time-homogeneous linear shot-noise with exponential attenuation function and exponentially distributed jumps is a decay-surge Markov process with triple

$$\left(\alpha(x) = -\alpha x;\ \beta(x) = \beta;\ k(x) = e^{-\theta x}\right).$$

Shot-noise processes being generically non-Markov, there is no systematic relationship of decay-surge Markov processes with shot-noise processes. In [14], it is pointed out that decay-surge Markov processes could be related to the maximal process of nonlinear shot noise; see Eliazar and Klafter (2007 and 2009).

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References

[1] Adke, S. R. (1993). Records generated by Markov sequences. *Statistics & Probability Letters* **18,** 257-263.

[2] Asmussen, S., Kella, O. (1996). Rate modulation in dams and ruin problems. *J. Appl. Probab.* **33,** no. 2, 523-535.

[3] Boxma, O.; Kella, O.; Perry, D. (2011). On some tractable growth-collapse processes with renewal collapse epochs. *J. Appl. Probab.* **48A,** New frontiers in applied probability: a Festschrift for Soren Asmussen, 217-234.

[4] Boxma, O., Perry, D., Stadje, W. and Zacks, S. (2006). A Markovian growth-collapse model. *Adv. in Appl. Probab.* **38,** no. 1, 221-243.

[5] Brémaud, P. and Massoulié, L. (1996). Stability of nonlinear Hawkes processes. *The Annals of Probability* **24,** 1563-1588.

[6] Brockwell, P.J, Resnick, S. I. and Tweedie, R. L. (1982). Storage processes with general release rule and additive inputs. *Adv. in Appl. Probab.* **14,** no. 2, 392-433.

[7] Carlson, J.M., Langer J.S. and Shaw B.E. (1994). Dynamics of earthquake faults *Rev. Mod. Phys.* **66,** 657.

[8] Çinlar, E. and Pinsky, M. (1971). A stochastic integral in storage theory. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **17,** 227-240.

[9] Cohen J. W. (1982) *The Single Server Queue* (North-Holland, Amsterdam).

[10] Davis, M. H. A. (1984). Piecewise-deterministic Markov processes: A general class of non-diffusion. *J. R. Statist. Soc. B* **46,** No.3, pp. 353-388.

[11] Daw, A. and Pender, J. (2019). Matrix Calculations for Moments of Markov Processes. arXiv:1909.03320.

[12] Eliazar, I. and Klafter, J. (2006). Growth-collapse and decay-surge evolutions, and geometric Langevin equations. *Physica A* **387,** 106-128.

[13] Eliazar, I. and Klafter, J. (2007) Nonlinear Shot Noise: From aggregate dynamics to maximal dynamics. *Euro-Physics Letters.* **78,** 40001.

[14] Eliazar, I. and Klafter, J. (2009). The maximal process of nonlinear shot noise. *Physica A* **388,** 1755-1779.

[15] Goncalves, B., Huillet, T. and Löcherbach, E. (2022). On population growth with catastrophes. *Stochastic Models* **38:2,** 214–249.

[16] Goncalves, B., Huillet, T. and Löcherbach, E. (2020). On decay-surge population models. arXiv:2012.00716v1.

[17] Gripenberg, G. (1983). A stationary distribution for the growth of a population subject to random catastrophes. *J. Math. Biol.* **17,** no. 3, 371-379.

[18] Hanson, F. B. and Tuckwell, H. C. (1978). Persistence times of populations with large random fluctuations. *Theoret. Population Biol.* 14, 46-61.

[19] Harrison, J. M. and Resnick, S. I. (1976). The stationary distribution and first exit probabilities of a storage process with general release rule. *Math. Oper. Res.* **1,** no. 4, 347-358.

[20] Harrison, J. M. and Resnick, S. I. (1978). The recurrence classification of risk and storage processes. *Math. Oper. Res.* **3,** no. 1, 57-66.

[21] Has'minskii, R.Z. (1980) *Stochastic Stability of Differential Equations*, Sijthoff and Noordhoff, Aalphen.

[22] Hawkes, A. G. (1971). Spectra of some self-exciting and mutually exciting point processes. *Biometrika* **58,** 83–90.

[23] Huillet, T. (2021). A shot-noise approach to decay-surge population models. Preprint, https://hal.archives-ouvertes.fr/hal-03138215.

[24] Kella, O. (2009). On growth-collapse processes with stationary structure and their shot-noise counterparts. *J. Appl. Probab.* **46,** no. 2, 363-371.

[25] Kella O. and Stadje W. (2001). On hitting times for compound Poisson dams with exponential jumps and linear release. *Journal of Applied Probability.* **38,** 781-786.

[26] Malrieu, F. (2015). Some simple but challenging Markov processes. *Annales de la Faculté des Sciences de Toulouse.* **24,** No. 4, 857-883.

[27] Meyn, S. and Tweedie, R. (1993). Stability of Markovian processes III: Foster-Lyapunov criteria for continuous-time processes. *Advances in Applied Probability.* **25,** No. 3, 518-548.

[28] Parzen E. (1999). *Stochastic Processes.* (Classics in applied mathematics, Vol. 24) SIAM, Philadelphia.

[29] Privault, N. (2020). Recursive computation of the Hawkes cumulants. arXiv:2012.07256.

[30] Privault, N. (2021). Moments of Markovian growth-collapse processes. arXiv:2103.04644.

[31] Richetti P., Drummond C., Israelachvili J. and Zana R. (2001). Inverted stick-slip friction. *Europhys. Lett.* **55,** 653.

[32] Snyder, D. L. and Miller, M. I. (1991). *Random Point Processes in Time and Space.* Second Edition. Springer-Verlag.