On some intriguing properties of the absorption time of a class of self-similar processes

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Joint work with A. Srapyonian, and, R. Loeffen and M. Savov

ECODEP CONFERENCE

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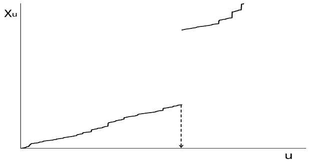

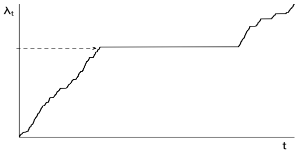

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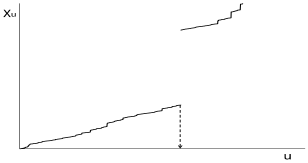

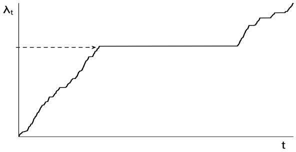

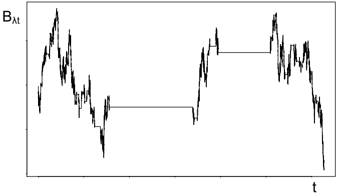

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⇝ Dynamics of complex systems with memory effects

⇝ Application in physics, neurology, epidemiology, pollution dynamics, biochemical reaction, mathematical finance, ruin theory ...

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$\rightsquigarrow$ $B_{\mathfrak{c}}$ is not self-similar anymore

(except when B is a stable Lévy process and $\mathfrak{c}$ is inverse stable)

Definition

Let $\mathbb{X}$ be the stochastic process defined, for $t \geq 0$, by

$$\mathbb{X}_t = X_{\mathfrak{c}_t} \quad \text{where } \mathfrak{c}_t = \inf\{s > 0; \chi_s > t\}$$

where X and χ are taken independent and are such that

- X is a α-self-similar Markov process ($\mathrm{S}_\alpha\mathrm{S}$) issued from $x > 0$
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Note that $\mathbb{T}$ is also the absorption time of $\mathbb{X}^+ = (\mathbb{X}_t, 0 \leq t < \mathbb{T})$ with $\mathbb{X}^+$ being a positive $\mathbb{S}_{\frac{\alpha}{\beta}}\mathbb{S}$ issued from $x > 0$

$\mathbb{T} = \inf\{t > 0;\ \mathbb{X}_t \leq 0\}$ where $\mathbb{X}_t = X_{\mathfrak{c}_t}$

- ◇ $\mathfrak{c}_t = \chi_t = t$, i.e. $\mathbb{X} = X$ is SαS and $\mathbb{T} = \mathrm{T}$
  - BERTOIN AND YOR (09) for X with no negative jumps
  - DALANG AND PESKIR (12), KUZNETSOV (13) for X a stable Lévy process
  - P. (14) for X has no positive jumps
  - P. AND SAVOV (21) for any X

- ◇ $\mathbb{X}_t = Z_{\mathfrak{c}_t} - \tau_t$, Z a Lévy process, $\mathfrak{c}$ the inverse of a subordinator χ, and τ is another subordinator: CONSTANTINESCU, LOEFFEN AND P. (23)

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  where $-\Psi(0), \sigma^2 \geq 0, \mathrm{a} \in \mathbb{R}$ and M a Lévy measure$\}$
- $\mathcal{B} = \{\phi(z) = \phi(0) + \mathrm{d}z + \int_0^\infty 1 - e^{-zy} m(dy), \phi(0), \mathrm{d} \geq 0, m$ a Lévy measure$\}$

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PROPOSITION 1

$$\mathbb{X} = X_{\mathbb{c}} \in \mathbb{S}_1\mathbb{S} \overset{\text{LAMPERTI}}{\leftrightarrow} (\Psi, \phi) \in \mathcal{N} \times \mathcal{B} \overset{\text{WH}}{\leftrightarrow} (\phi_+, \phi_-, \phi) \in \mathcal{B}^2 \times \mathcal{B}_{\mathbb{c}}$$

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$$\mathbb{T} = \mathbb{T}_\Psi(\phi) = \inf\{t > 0;\ \mathbb{X}_t \leq 0\} \neq \infty \iff \phi_+(0) > 0$$

For any $\phi \in \mathcal{B}$, the Bernstein-gamma function W_ϕ is the unique positive-definite function solution to

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1) For any $-a < \mathfrak{Re}(z) < b$, $a, b > 0$ explicit,

$$\mathbb{E}_x\left[\mathbb{T}_\Psi^z(\phi)\right] = x^z \frac{\phi_+(0)}{\phi'(0)} \frac{\Gamma(-z)}{W_\phi(-z)} \frac{\Gamma(z+1) W_{\phi_+}(-z)}{W_{\phi_-}(z+1)}$$

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Characterization and properties of $\mathbb{T}_\Psi(\phi)$

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$f_{\mathbb{T}_\Psi(\phi)}$ is smoother than f_{T_Ψ} by $\lceil \mathrm{N} \rceil - \lceil \mathrm{N}_\Psi \rceil > 0$ number of derivatives

Some intriguing identities I

From P., we have, for any $\phi \in \mathcal{B}_{-1} = \{\phi \in \mathcal{B};\, 0 \leq \phi(u) < \infty \; u \geq -1\}$,

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where $\mathbb{P}(\mathbb{F}(\phi) \in dt) = t^{-2}\mathrm{I}_\phi(t^{-1})dt$, $\mathrm{I}_\phi(z) = \sum_{n=0}^{\infty} \frac{n+1}{\phi(n+1)} \frac{(-z)^n}{W_\phi(n+1)}$

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2) If $\Psi \in \mathcal{N}_{-1}^{+} = \{\Psi \in \mathcal{N}_1 \text{ with } \phi_-(u) = u\}$, i.e. $\mathbb{X}$ has no negative jumps,

$$\mathbb{T}_\Psi(\mathcal{S}\phi_+) \stackrel{d}{=} x\mathbb{F}$$

where $\mathbb{P}(\mathbb{F} \in dt) = t^{-2}e^{-t^{-1}}dt$ is the Fréchet distribution

Some intriguing identities I

From P., we have, for any $\phi \in \mathcal{B}_{-1} = \{\phi \in \mathcal{B};\ 0 \leq \phi(u) < \infty\ u \geq -1\}$,

$$\mathcal{S}\phi(u) = \frac{u}{u+1}\phi(u) \in \mathcal{B}$$

Theorem

1) For any $\Psi \in \mathcal{N}_{-1} = \{\Psi = -\phi_-\phi_+ \in \mathcal{N} \text{ with } \phi_+ \in \mathcal{B}_{-1}\}$,

$$\mathbb{T}_\Psi(\mathcal{S}\phi_+) \stackrel{d}{=} x\mathbb{F}(\phi_-)$$

where $\mathbb{P}(\mathbb{F}(\phi) \in dt) = t^{-2}\mathrm{I}_\phi(t^{-1})dt$, $\mathrm{I}_\phi(z) = \sum_{n=0}^{\infty} \frac{n+1}{\phi(n+1)} \frac{(-z)^n}{W_\phi(n+1)}$

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For any $\phi_- \in \mathcal{B}$, $(\mathbb{T}_{\phi_+,\phi_-}(\mathcal{S}\phi_+))_{\phi_+ \in \mathcal{B}}$ have the same distribution

Some intriguing identities II

Recall that, for any $\phi \in \mathcal{B}_{-1} = \{\phi \in \mathcal{B};\ 0 \leq \phi(u) < \infty \text{ for all } u \geq -1\}$,

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Theorem

For all $\Psi \in \mathcal{N}_{-1} = \{\Psi = -\phi_-\phi_+ \in \mathcal{N} \text{ with } \phi_+ \in \mathcal{B}_1\}$ we have

$$\mathbb{T}_\Psi(\mathcal{S}\phi_+) \stackrel{d}{=} \mathrm{T}_\psi$$

where $\mathrm{T}_\psi = \inf\{t > 0;\ \overline{X}_t \leq 0\}$ with $\overline{X}$ a $\mathrm{S_1S}$ with no positive jumps associated, via the Lamperti mapping, to $\psi(z) = (z-1)\phi_-(z) \in \mathcal{N}$.

Recall that, for any $\phi \in \mathcal{B}_{-1} = \{\phi \in \mathcal{B};\ 0 \leq \phi(u) < \infty \text{ for all } u \geq -1\}$,

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Interpretation that the time-change annihilates the upward jumps of $\mathbb{X}$

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Interpretation that the time-change annihilates the upward jumps of $\mathbb{X}$

FPT do not characterize a process