Construction, consistency, asymptotic normality of viscosity estimation for 2D Navier-Stokes

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EcoDep conference, 1 oct 2024

1 Introduction

2 Main results
- Consistency
- Asymptotic normality

3 Schema of proof
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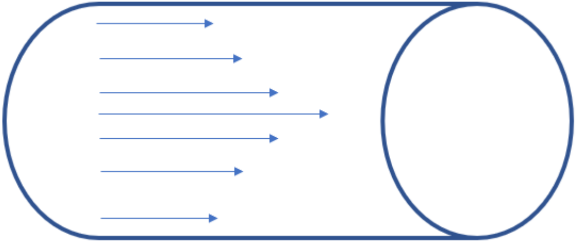
Introduction

- fluid friction, both within the fluid itself and between the fluid and its surroundings
- Juice has **low** viscosity.
- Syrup has **high** viscosity.

Fluid flow in pipes



- all the parts of the fluid do not move with equal velocity
- friction between the fluid and the pipe
- friction between fluid layers
- the fluid's resistance to flow is known as **viscosity**.

Application of measurement of viscosity of fluid

Measurement of viscosity plays an important role in the quality control and various research and development stages

- environmental, mechanical, and bio-mechanical engineering
- oil production

The Navier–Stokes equations are partial differential equations which describe the motion of viscous fluid substances. Mathematically, we use 3D Navier–Stokes system in a cylindrical domain

- Difficulties mathematical investigation
  - the absence of global well-posedness of the PDE
  - the unboundedness of the physical domain

Therefore, the problem is studied in a two-dimensional strip, assuming the periodicity in the unbounded direction and no-slip condition.

Model and notations

We consider the 2D Navier-Stokes system in

$$D = \{x = (x_1, x_2) \in \mathbb{R}^2 : -1 < x_2 < 1\},$$

$$\partial_t u + \langle u, \nabla \rangle u - \nu \Delta u + \nabla p = \eta(t,x), \quad \operatorname{div} u = 0, \tag{1.1}$$

$$u\big|_{x_2=\pm 1} = 0, \quad \theta_a u \equiv u. \tag{1.2}$$

Here $\nu > 0$ is the viscosity

$$\eta(t,x) = \frac{\partial}{\partial t}\zeta(t,x), \quad \zeta(t,x) = \sum_{j=1}^{\infty} b_j \beta_j(t) e_j(x), \tag{1.3}$$

where

- $\{e_j\}$ is an orthonormal basis of eigenfunctions of Stokes operator
- $\{\beta_j\}$ are independent standard Brownian motions
- $B := \sum_{j=1}^{\infty} b_j^2 > 0$

Construction $\hat{\nu}_t$

Let $H = \{u \in L^2(D, \mathbb{R}^2), \mathrm{div} u = 0\}$, and $\Pi : L^2(D, \mathbb{R}^2) \longrightarrow H$ the orthogonal projection. Applying Π to the Navier–Stokes system (1.1)

$$\partial_t u + \nu L u + \Pi(\langle u, \nabla \rangle u) = \eta(t), \tag{1.4}$$

Itô formula to $\|u\|^2$, we derive

$$\|u(t)\|^2 + 2\nu \int_0^t \|\nabla u(s)\|^2 \mathrm{d}s = \|u_0\|^2 + Bt + 2\int_0^t (u(s), \mathrm{d}\zeta(s)),$$

This equivalent to

$$\xi_t := \frac{1}{t}\int_0^t \|\nabla u(s)\|^2 \mathrm{d}s = \frac{B}{2\nu} + \frac{1}{2\nu t}\left(\|u_0\|^2 - \|u(t)\|^2\right) + \frac{1}{\nu t}\int_0^t (u(s), \mathrm{d}\zeta(s)).$$

Let $t \to \infty$, it is natural to define an estimator for ν by the formula

$$\hat{\nu}_t = \frac{B}{2\xi_t}$$

Estimator is well-defined

Theorem

With probability 1, *the estimator* $\hat{\nu}_t = \frac{B}{2\xi_t}$ *is well defined.*

Proof.

Since $\mathbb{P}\big(\{u \in \mathcal{H}\}\big) = 1$, it suffices to prove that

$$\mathbb{P}\big(\{\xi_t = 0\} \cap \{u \in \mathcal{H}\}\big) = 0. \tag{1.5}$$

Let $\omega \in \Omega$ be such that $u \in \mathcal{H}$ and $\xi_t = 0$. Then $\nabla u(s) = 0$ for almost every $s \in [0, t]$. Since $u(s) \in V$ a.e and u is a continuous function of time with range in H, we conclude that $u(s) = 0$ for $s \in [0, t]$. It follows from (1.4) that $\eta(s) = 0$ for $s \in [0, t]$. Since $B > 0$, there is an integer $j \geq 1$ such that $\beta_j(s) = 0$ for $s \in [0, t]$. This event has probability zero, so that we arrive at (1.5). □

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Consistency

Theorem

$$\mathbb{P}_u\Big\{\hat{\nu}_t \to \nu \text{ as } t \to \infty\Big\} = 1$$

Furthermore, for any $\nu \in (0,1]$ *and* $\varepsilon \in (0, \frac{1}{2})$*, there is a random time* $T \geq 1$ *such that*

$$|\hat{\nu}_t - \nu| \leq t^{-\frac{1}{2}+\varepsilon} \quad \text{for } t \geq T. \tag{2.1}$$

Asymptotic normality

Theorem

For any $\nu > 0$ *there is* $\sigma_\nu > 0$ *such that*

$$\mathcal{D}\big(\sqrt{t}(\hat{\nu}_t - \nu)\big) \rightharpoonup \mathcal{N}_{\sigma_\nu} \quad \text{as } t \to \infty \text{ for any } u \in H,$$

Furthermore,

$$\sup_{z \in \mathbb{R}} \left| \mathbb{P}_\lambda \left\{ \sqrt{t}\,(\hat{\nu}_t - \nu) \le z \right\} - \Phi_{\sigma_\nu}(z) \right| \le \text{const}\, t^{-\frac{1}{4}+\varepsilon}, \qquad (2.2)$$

where $\mathcal{N}_\sigma \in \mathcal{P}(\mathbb{R})$ be the centered normal law with variance $\sigma^2 > 0$ and Φ_σ its distribution.

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Consistency

Instead of $\hat{\nu}_t$, we work with ξ_t

$$|\hat{\nu}_t - \nu| = \frac{\nu}{\xi_t}\left|\xi_t - \frac{B}{2\nu}\right|. \tag{3.1}$$

Recall that

$$\xi_t = \frac{1}{t}\int_0^t \|\nabla u(s)\|^2 \mathrm{d}s = \frac{B}{2\nu} + \frac{1}{2\nu t}\left(\|u_0\|^2 - \|u(t)\|^2\right) + \frac{1}{\nu t}\int_0^t (u(s), \mathrm{d}\zeta(s)).$$

We can rewrite

$$\left|\xi_t - \frac{B}{2\nu}\right| \leq \boxed{\frac{1}{2\nu t}\|u_0\|^2} + \boxed{\frac{1}{2\nu t}\|u(t)\|^2} + \boxed{\frac{1}{\nu t}|M_t|}.$$

here $M_t = \int_0^t (u(s), \mathrm{d}\zeta(s))$. We estimate 3 terms

Proof of consistency

Three terms

1. $\frac{1}{2\nu t}\|u_0\|^2 \leq \frac{1}{3}\, t^{-\frac{1}{2}+\epsilon}$ for $t \geq T_1$ holds for $T_1 = \frac{9}{4\nu^2}\|u_0\|^4$.

2. $\frac{1}{2\nu t}\|u(t)\|^2 \leq \frac{1}{3}\, t^{-\frac{1}{2}+\epsilon}$ for $t \geq T_2$ holds for

$$T_2 = \min\Big\{N \geq 1 : \sup_{k\leq t\leq k+1} \|u(t)\|^2 \leq \frac{2\nu}{3}\, k^{1/2} \text{ for } k \geq N\Big\}$$

3. $\frac{1}{\nu t}|M_t| \leq \frac{1}{3}\, t^{-1/2+\varepsilon}$ for $t \geq T_3$ holds for

$$T_3 = \min\{N \geq 1 : |M_k| \leq \frac{\nu}{10}\, k^{\frac{1}{2}+\varepsilon}\}$$

Asymptotic normality

$$\sup_{z\in\mathbb{R}} \left|\mathbb{P}_\lambda\left\{\sqrt{t}\left(\hat{\nu}_t - \nu\right) \le z\right\} - \Phi_{\sigma_\nu}(z)\right| \le \text{const } t^{-\frac{1}{4}+\varepsilon}, \qquad (3.2)$$

Some useful tools

- Recall that $\hat{\nu}_t = \dfrac{B}{2\xi_t}$, then we must prove that

$$\boxed{\sup_{z\in\mathbb{R}} \left|\mathbb{P}_\lambda\left\{\sqrt{t}\left(\xi_t - (2\nu)^{-1}B\right) \le z\right\} - \Phi_{\sigma_\nu}(z)\right| \le \text{const } t^{-\frac{1}{4}+\varepsilon},} \qquad (3.3)$$

- Recall that

$$\left|\xi_t - \frac{B}{2\nu}\right| \le \frac{1}{2\nu t}\|u_0\|^2 + \frac{1}{2\nu t}\|u(t)\|^2 + \frac{1}{\nu t}|M_t|.$$

Idea of poof (cont)

Denote $\Delta_\sigma(X,z) = \mathbb{P}\{X \le z\} - \Phi_\sigma(z)$, then

$$\boxed{\sup_{z\in\mathbb{R}} |\Delta_\sigma(X,z)| \le \sup_{z\in\mathbb{R}} |\Delta_\sigma(Y,z)| + \mathbb{P}\{|X-Y| > \varepsilon\} + c_\sigma\varepsilon,}$$

- Using this property with $X = \sqrt{t}\left(\xi_t - (2\nu)^{-1}B\right)$ and $Y = k^{-1/2}M_k$, where k is the integer part of t.

$$\sup_{z\in\mathbb{R}} \left|\mathbb{P}_\lambda\{k^{-1/2}M_k \le z\} - \Phi_\sigma(z)\right| \le \text{ const } k^{-\frac{1}{4}+\varepsilon}$$

- Law of large numbers for conditional variance of zero-mean martingale, it suffices to establish for θ and Θ,

$$\mathbb{E}_\lambda \exp(\theta\,|M_k - M_{k-1}|) \le \Theta \quad \text{for } k \ge 1. \qquad (3.4)$$

It is done by properties of martingale and some technique calculation in using Cauchy-Swharz inequality.

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