# Modeling abundance time series through a «pseudo-HMM» framework

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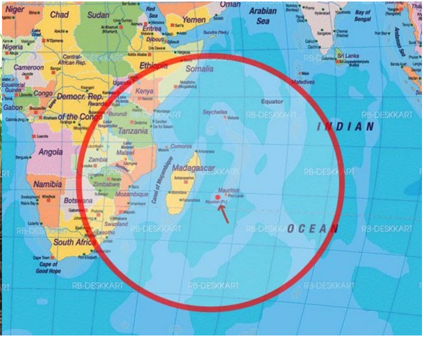
# Outlines

# Insect pests

◎ The objective of the initial study is to control a population of insect pests in a sugar-cane field in La Réunion, while limiting the use of pesticides.



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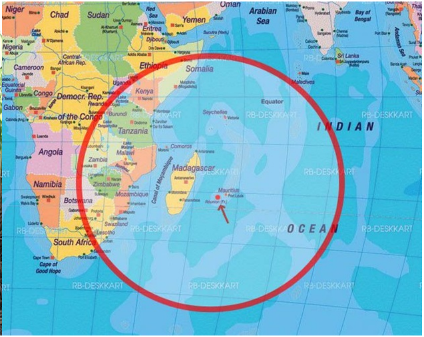


It requires a good understanding of the ecosystem, and the impacts of species on each other.

# The study (1/2)

We focus here on four group of species

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(b) Diptera

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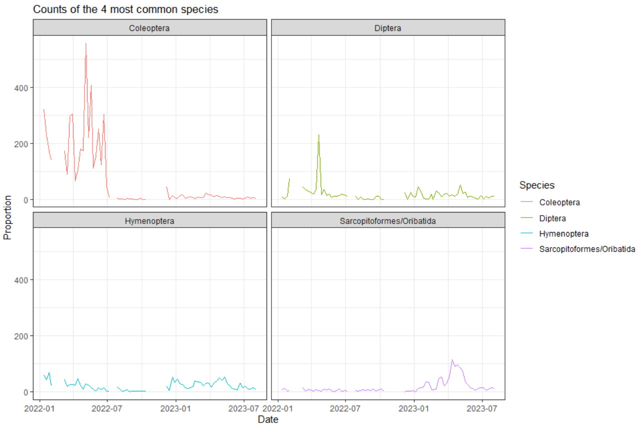
Their population has been observed weekly from January, 2022 to August, 2023, catching the insects into traps.

# The study (2/2)

We thus obtain a time series of counts $(Y_t)_{1\leqslant t\leqslant 82}$ valued in $\mathbb{N}^4$

$$Y_t = (Y_{1,t},\ Y_{2,t},\ Y_{3,t},\ Y_{4,t}),\ 1 \leqslant t \leqslant 82.$$



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$X_t$ is the vector of proportions of each species in the whole ecosystem. For $p$ species, $X_t$ is valued in the simplex

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Our goal is to provide a model for the joint process $(X_t, Y_t)_{t\in\mathbb{Z}}$, where $(X_t)_{t\in\mathbb{Z}}$ is not observed.

# Outlines

# Dynamic of the relative abundance $(X_t)_{t\in\mathbb{Z}}$ (1/2)

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- ➜ $\varphi_t \in \mathbb{R}_+^*$ a dispersion parameter

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- ➜ $\mu_t = (\mu_{1,t}, \ldots, \mu_{p,t}) \in \mathcal{S}_{p-1}$ a mean parameter

  $$\log\left(\frac{\mu_{i,t}}{\mu_{p,t}}\right) = A_0(i) + A_1(i,\cdot)\cdot(X_{1,t-1}, \ldots X_{p-1,t-1})' + B(i,\cdot)\cdot Z_t,\ 1 \leqslant i \leqslant p-1,$$

  where $A_0 \in \mathbb{R}^{p-1}$, $A_1$ is a matrix of dimension $(p-1)\times(p-1)$ and $B$ is a matrix with $(p-1)$ rows.

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$$\forall x \in \mathcal{S}_{p-1},\ \forall z \in \mathbb{R}^k,\ \forall A \in \mathcal{B}\left(\mathcal{S}_{p-1}\right)\ \ P(A \mid x, z) \geqslant \varepsilon(z)\mu(z, A),$$

for some fixed measurable application $\varepsilon$ valued in $(0, 1]$ and a fixed Markov kernel $\mu$ defined on $\mathbb{R}^k \times \mathcal{B}(\mathcal{S}_{p-1})$.

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Moreover, if $(Z_t)_{t\in\mathbb{Z}}$ is ergodic, then $(X_t, Z_t)_{t\in\mathbb{Z}}$ is also ergodic.

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Dirichlet density function:

$$f_\alpha(x_1, \dots, x_p) = \frac{\Gamma(\sum \alpha_i)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_p)} x_1^{\alpha_1 - 1} \cdots x_p^{\alpha_p - 1}.$$

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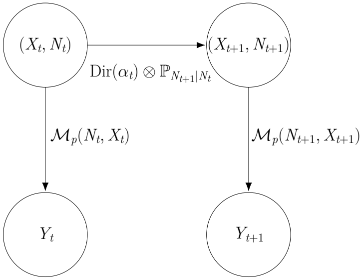


We thus obtain a «pseudo-HMM» framework, where the underlying process is partially observed.

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# Outlines

# Joint log-likelihood

Assume that for some $T \in \mathbb{N}^*, (y_0, n_0), \ldots, (y_T, n_T)$ are observed realizations of our process, corresponding to the unobserved relative abundances $x_0, \ldots, x_T$.

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The joint log-likelihood of our process is

$$\mathcal{L}_\theta(x_{0:T}, y_{0:T}) = \sum_{t=1}^{T} \Bigg\{ \log(n_t!) + \log(\Gamma(\phi_t)) + \sum_{j=1}^{p} \left[ (\alpha_{j,t} + y_{j,t} - 1) \log(x_{j,t}) - \log(y_{j,t}!) - \log(\Gamma(\alpha_{j,t})) \right] \Bigg\} + \log(n_0!) + \sum_{j=1}^{p} \left[ y_{j,0} \log(x_{j,0}) - \log(y_{j,0}!) \right].$$

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$$\left.\left. - \log(\Gamma(\alpha_{j,t})) \right] \right\} + \log(n_0!) + \sum_{j=1}^{p} \left[ y_{j,0}\log(x_{j,0}) - \log(y_{j,0}!) \right].$$

We are interested in the estimation of the true parameter $\theta_0$ of our model, where

$$\theta = (a_0, a_1, A_0(1), \dots, A_0(p-1), A_1(1,1), \dots, A_1(p-1, p-1), B(1,1), \dots B(k, p-1)).$$

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✗ The quantity $\mathcal{I}\left(\theta^{(n)}, \theta\right)$ can not by computed directly.

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The idea is to perform a large number $N$ of simulated trajectories $\widetilde{X}_{0:T}^{(1)}, \ldots, \widetilde{X}_{0:T}^{(N)}$, where for each $1 \leqslant i \leqslant N$

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We then approach $\mathcal{I}\left(\theta^{(n)}, \theta\right)$ by the mean

$$\sum_{i=1}^{N} w_i \mathcal{L}_\theta(\widetilde{X}^{(i)}_{0:T}, y_{0:T}).$$

# About the particle filter (1/2)

The proposal distribution chosen here is the one of a Markov chain $\left(\widetilde{X}_t\right)_{t\in\mathbb{Z}}$ with transition kernel

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Resampling strategy:

- Each simulation $\widetilde{X}_{0:T}^{(i)}$ and weight $w_i$ are computed sequentially.
- Once the **particles** $\widetilde{X}_t^{(1)}, \ldots, \widetilde{X}_t^{(N)}$ and the weights $w_{t,1}, \ldots, w_{t,N}$ are computed, we replace the particles by a new sample, drawn with replacement, from the same set of particles, with probabilities given by the particles weights.

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Given the weighted set of particles $\left\{\widetilde{X}_t^{(i)}, w_{t,i} : 0 \leqslant t \leqslant T, 1 \leqslant i \leqslant N\right\}$:

- Set $\xi_T = \widetilde{X}_T^{(i)}$ with probability $w_{i,T}$.

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The trajectory $\xi_{0:T}$ obtained is distributed with respect to $\mathbb{P}_{\theta^{(n)}}(\cdot \mid y_{0:T})$.

# Outlines

I. The problematic

II. The framework

III. Estimation procedure

IV. Numerical experiments

## Simulations (1/3)

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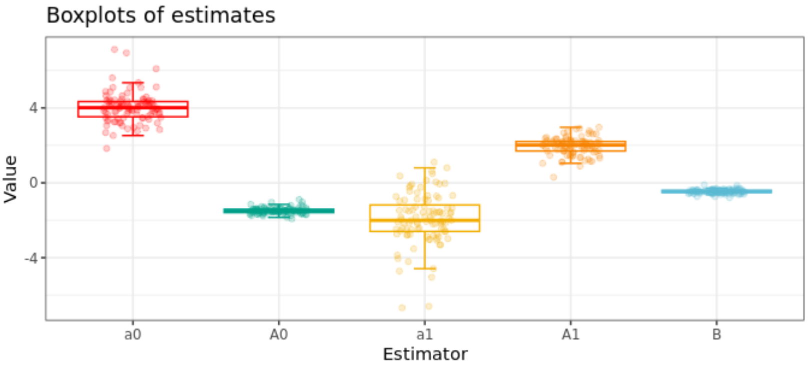
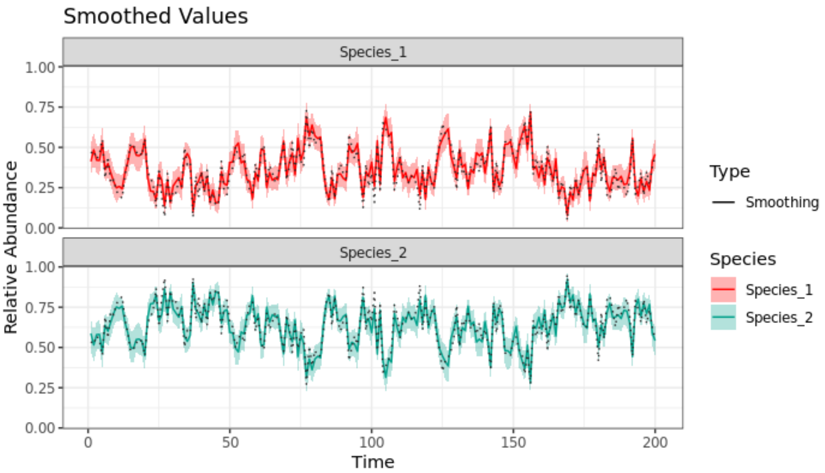
| Parameter | True Value | Mean Estimate | MSE | Bias | Variance |
|---|---|---|---|---|---|
| $a_0$ | 4 | 4.019 | 0.629 | 0.019 | 0.628 |
| $a_1$ | -2 | -1.932 | 1.872 | 0.068 | 1.868 |
| $A_0$ | -1.5 | -1.490 | 0.026 | 0.010 | 0.026 |
| $A_1$ | 2 | 1.959 | 0.220 | -0.041 | 0.219 |
| $B$ | -0.5 | -0.466 | 0.127 | 0.034 | 0.012 |

# Simulations (2/3)



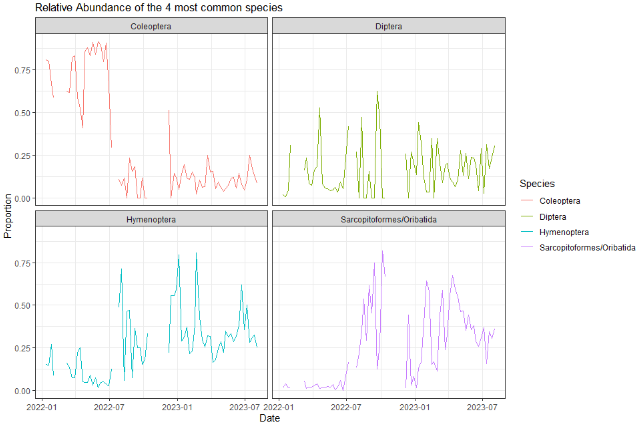
# Simulations (3/3)

Once we obtain estimates of the model's parameters, it is possible to use backward smoothing to recover the hidden process of relative abundance.



# Back in La Réunion (1/2)

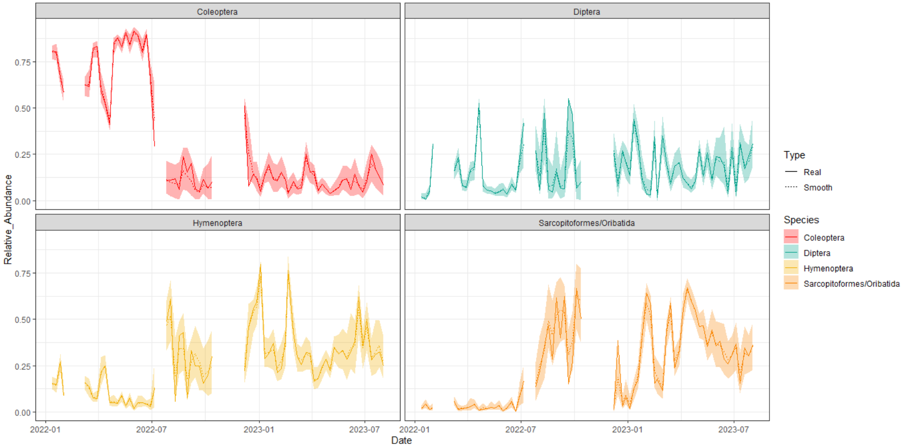
We finally fit our model to the relative abundance of the insects in a sugar-cane field in La Réunion.




We use the temperature and the amount of precipitation as exogenous variables.

# Back in La Réunion (2/2)

Once our estimation is complete, we use backward smoothing in order to recover the correct relative abundance of our species.



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- Add a variable selection.

# Thank you !