Hawkes processes with random environment

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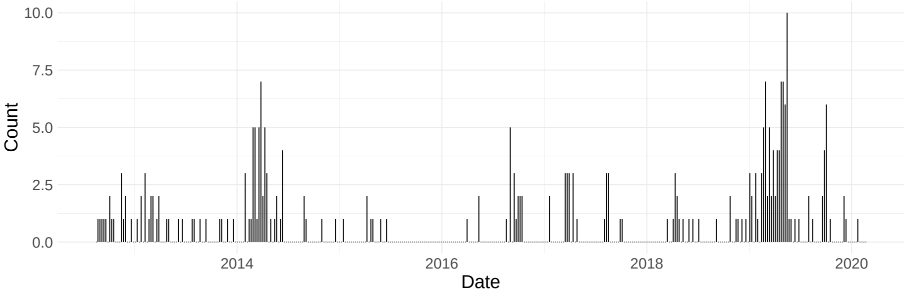
Context: modeling epidemiological data

Weekly count of measles cases in the prefecture of Tokyo.

- Highly contagious viral disease, transmitting via droplets.
- Sprung back through imported cases and non-vaccinated individuals.
- Notifiable disease: 264 cases in 8 years.



Outline

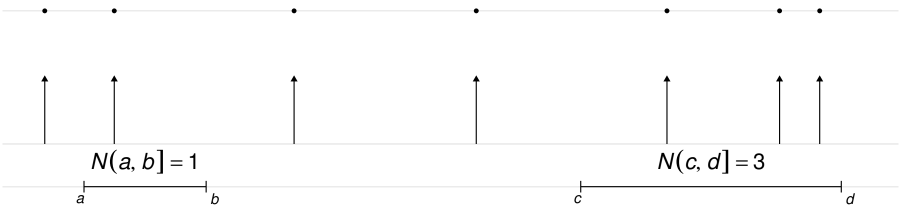
Point process

Definition: Point process N on $\mathbb{R}$

Measurable map N:

$$\begin{aligned} N : (\Omega, \mathcal{F}, \mathbb{P}) &\to (\mathfrak{N}, \mathcal{N}) \\ \omega &\mapsto N(\omega, \cdot) \end{aligned}$$

where $\mathfrak{N}$ is the set of locally finite counting measures on $\mathbb{R}$.



Hawkes process

Conditional intensity λ of point process N

$\lambda(t)dt$ is the conditional probability that there will be an atom of N between t and $t + dt$, given the realisations of N before t:

$$\lambda(t)dt = \mathbb{P}(N(dt) > 0 \mid \{t_j\}, t_j < t)$$

Linear Hawkes process on the real half-line (Hawkes, 1971)

Self-exciting point process defined by its conditional intensity function:

$$\lambda(t) = \eta + \mu \int_{-\infty}^{t} h(t-s)N(\mathrm{d}s) = \eta + \mu \sum_{t_j < t} h(t - t_j)$$

where $\eta > 0$, $\mu \in (0,1)$, h is an integrable nonnegative function such that $\int_{\mathbb{R}+} h = 1$, and $(t_j)_{j\in\mathbb{N}}$ are realisations of the point process.

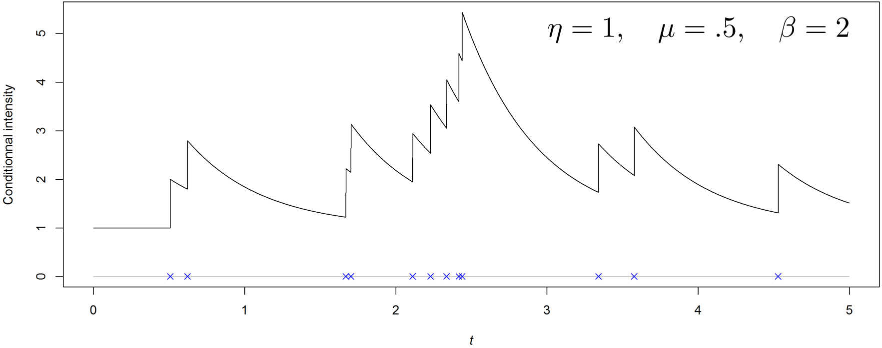
The occurrence of any event increases temporarily the probability of further events occurring.

Hawkes process

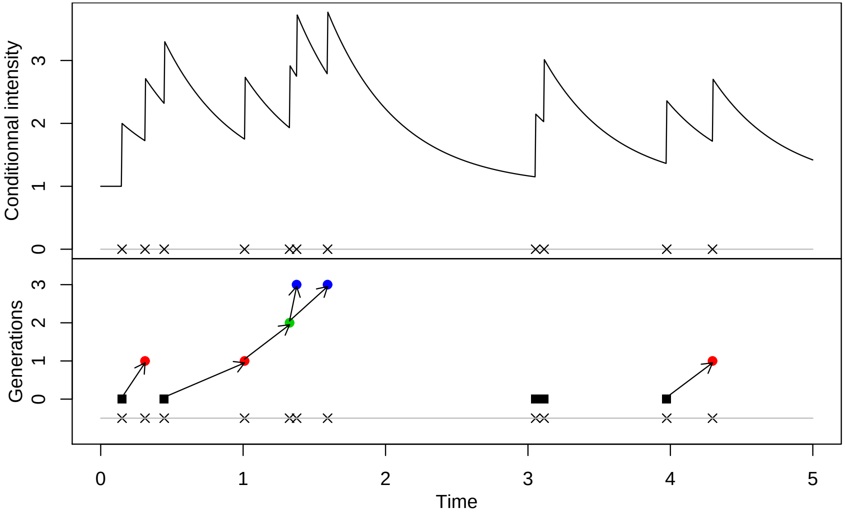
Linear Hawkes process on the real half-line

With exponentially decaying intensity:

$$\lambda(t) = \eta + \mu \sum_{t_j < t} \beta e^{-\beta(t-t_j)}$$



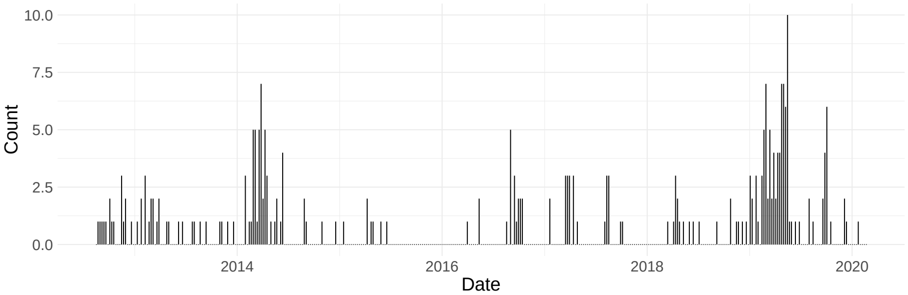
Hawkes process as a branching process



In the literature

- Self-exciting and clustering properties.
- Many disciplines of application:
  - seismology (Adamopoulos, 1976), neurophysiology, finance (Bacry, Mastromatteo, and Muzy, 2015), genomics (Reynaud-Bouret and Schbath, 2010), epidemiology, etc.
  - general review (Reinhart, 2018).
- Interesting properties:
  - Poisson cluster process: each cluster is a continuous-time Galton-Watson tree (Hawkes and Oakes, 1974).
  - Martingale properties of $N(t) - \int_0^t \lambda(s)\mathrm{d}s$ and $\left(N(t) - \int_0^t \lambda(s)\mathrm{d}s\right)^2 - \int_0^t \lambda(s)\mathrm{d}s$.
  - Erlang kernel $\rightarrow$ piecewise deterministic Markov process (Duarte, Löcherbach, and Ost, 2019).

Case-study: transmission of Measles in Tokyo[1]



Gaussian reproduction kernel: $h(t) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(t-\nu)^2}{2\sigma^2}\right)$

- $\widehat{\nu} = 9.8$ days, $\widehat{\sigma} = 5.9$ days

Epidemiology (Centers for Disease Control and Prevention, 2015)

Incubation period: 10-12 days after exposure.

Transmission period: 4 days before to 4 days after rash onset.

[1] https://www.niid.go.jp/niid/en/survaillance-data-table-english.html

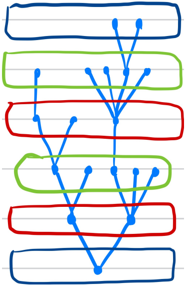
Our motivation

- In epidemiology, $\mu \sim \mathcal{R}_0$ (basic reproduction number).
- **Problem**: μ fixed, must be < 1 for existence of the process.
- Extension possible in the framework of local stationarity (Dahlhaus, 1996; Roueff, Sachs, and Sansonnet, 2016).
  - Strict upper limit $\sup_{t \in \mathbb{R}} \{\mu_t\} < 1$.

- Galton-Watson with random environment (Smith and Wilkinson, 1969; Athreya and Karlin, 1971).
  - Reproduction law different between each generation.
  - Extinction of the process almost sure if

$$\mathbb{E}\left[\log \mu(\omega)\right] \leq 0,$$

with $\mu(\omega)$ the mean reproduction at a given generation.



Hawkes process with random environment (HPRE)

- Usually, random environment with Hawkes process is meant as

$$\lambda(t) = \eta + \mu \int_0^t h(t-s)N(\mathrm{d}s) + \int_0^t \sigma(s)\mathrm{d}B_s.$$

  - CSBPs with random environments obtained as scaling limits of BPREs (Bansaye and Simatos, 2015; Bansaye, Caballero, and Méléard, 2019).
  - No direct dependency on further generations.
  - Strong assumptions of exponential kernel $h \equiv \exp$.

- Existing works (Dassios and Zhao, 2011; Lee, Lim, and Ong, 2016) as

$$\lambda(t) = \eta + \int_0^t X_s h(t-s)N(\mathrm{d}s).$$

  - No systematic characterisation of the process (existence, stationarity) when the reproduction number increases above 1.
  - Assumption of exponential kernel $h \equiv \exp$.

Existence of HPREs

Our framework for HPREs:

$$\lambda(t) = \eta + \int_0^t H(t-s, X_s) N(\mathrm{d}s)$$
$$= \eta + \int_0^t X_s h(t-s) N(\mathrm{d}s),$$

with $\eta > 0$, $\int h = 1$, and (X_t) a non-negative stochastic process.

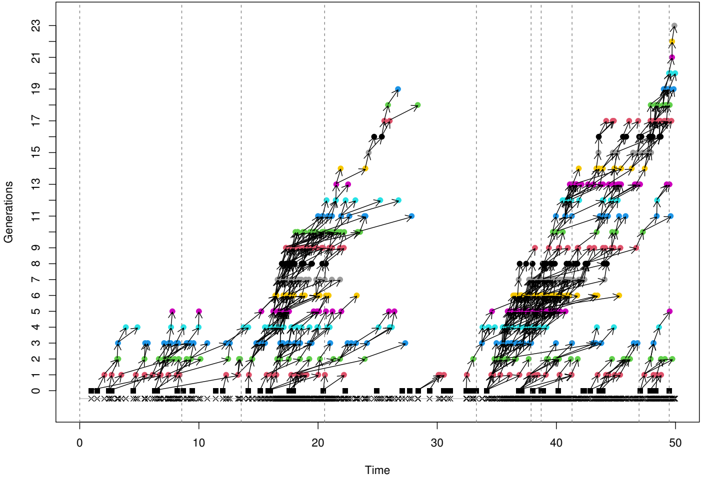
Proposition

Let $M > 0$, and define $\tau_M = \inf\{t > 0 : |\lambda(t)| + |X_t| \geq M\}$. Let Q denote the unitary bivariate Poisson point process. Then there exists a unique solution N with intensity λ over $[0, \tau_M]$ satisfying

$$\begin{cases} \lambda(t) = \eta + \int_0^t X_s h(t-s) N(\mathrm{d}s), \\ N\big([0,t]\big) = \int_0^t \int_0^\infty \mathbb{1}_{\{\theta \leq \lambda_s\}} Q(\mathrm{d}s, \mathrm{d}\theta). \end{cases}$$

A simulation of a HPRE

Random environment (X_t): alternating renewal process.



Discretisation scheme for HPRE

- Assume X_t satisfies the EDS

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t.$$

- Euler scheme X_t^n on $[0, T]$ with discretisation step T/n.
- HPRE N^n with intensity

$$\lambda_t^n = \lambda_0 + \int_{-\infty}^{t} X_s^n h(t-s) N_t^n(\mathrm{d}t).$$

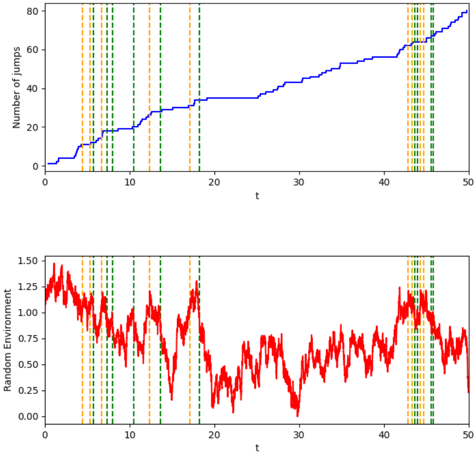
Proposition

Suppose that $b(\cdot)$ and $\sigma(\cdot)$ are globally Lipschitz. Then there exists an increasing function $K(\cdot)$ such that, for any $T > 0$ and $n \geq 1$,

$$\mathbb{P}\left(\int_0^T \left|N(\mathrm{d}t) - N^n(\mathrm{d}t)\right| \neq 0\right) \leq \frac{K(T)}{\sqrt{n}}$$

A more complex simulation of a HPRE

Random environment (X_t): Ornstein-Uhlenbeck.



Stationary distribution of the HPRE

Theorem (?)

Assume that the process (X_t) is stationary and ergodic, and that

$$\mathbb{E}\left[\log\left(\int_{\mathbb{R}} X_{-u} h(u) \mathrm{d}u\right)\right] < 0.$$

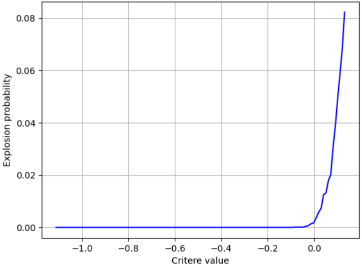
Then the branching process with random environment (X_t), starting at any location $s \in \mathbb{R}$, is subcritical, i.e. goes extinct almost surely.

- Proof based on branching process properties, similar to (Athreya and Karlin, 1971).
- Different from intuition: $\mathbb{E}\left[\log X_0\right] < 0$.
- Corollary: The HPRE exists and has a stationary distribution.

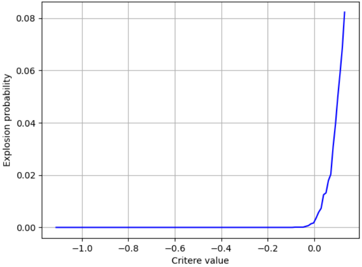
Simulation study for the critical condition

- **Objective**: Compute the probability of extinction of a single cluster started at 0.
- X_t Ornstein-Uhlenbeck simulated 1000 times.
  - Both criterions estimated by Monte-Carlo.
- 10 clusters simulated for each X_t.
  - Process is considered as non-extinct if $N([0, 1000]) > 10^6$.

$\mathbb{E}\left[\log\left(\int_{\mathbb{R}} X_{-u} h(u) \mathrm{d}u\right)\right]$



$\mathbb{E}[\log X_0]$



A lot of perspectives

- Correct the proof for the critical assumption of subcriticality.
- Methods for statistical inference of the HPRE.
- Spectral properties of the HPRE.
  - Adapted for unperfect data (agregated, missing, or noisy).

Thank you for your attention!

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