Moment conditions for random coefficient AR(∞) under non-negativity assumptions

Pascal Maillard and Olivier Wintenberger (alias the virus)

Feb 13, 2024: ECODEP, IHP

LPSM, Sorbonne University & WPI, University of Vienna

ECODEP motivations

Chain with infinite memory

Let (X_t) be a solution of the non-linear stochastic recurrent equation

$$X_t = F(X_{t-1}, X_{t-2}, \ldots ; \xi_t)\,, \qquad t \in \mathbb{Z}\,,$$

where (ξ_t) is iid.

Theorem (Doukhan and W., 2008)

Assume

$$\mathbb{E}[|F(x_0, x_1, x_2, \ldots ; \xi) - F(y_0, y_1, y_2, \ldots ; \xi)|] \leqslant \sum_{j=0}^{\infty} a_j |x_j - y_j|\,, \qquad (x_j)\,, (y_j) \in \mathbb{R}^{\mathbb{N}}\,,$$

$$\sum_{j=0}^{\infty} |a_j| < 1\,, \qquad \textit{and} \qquad \mathbb{E}[|F(0, 0, \ldots ; \xi)|] < \infty,$$

then the causal stationary solution (X_t) *exists and satisfies* $\mathbb{E}[|X|] < \infty$.

Application to Poisson Quasi Maximum Likelihood Estimation (PQMLE)

Poisson QMLE, Ahmad and Francq (2016)

PQMLE is defined under the assumption that we observe $X_t \in \mathbb{N}$, $t = 1, \ldots, n$ such that

$$\mathbb{E}[X_t | X_{t-1}, X_{t-2}, \ldots] = \lambda(X_{t-1}, X_{t-2}, \ldots; \theta_0) =: \lambda_t(\theta_0), \qquad a.s.,$$

for some parameter $\theta_0 \in \Theta$ and some measurable function λ.

Note that if (X_t) is a chain with infinite memory then

$$\mathbb{E}[X_t | X_{t-1}, X_{t-2}, \ldots] = \mathbb{E}_\xi[F(X_{t-1}, X_{t-2}, \ldots; \xi)], \qquad a.s.,$$

and Poisson QMLE corresponds to the parametrization of $F = F_{\theta_0}$, $\theta_0 \in \Theta$ (Bardet and W., 2009).

The asymptotic normality of PQMLE is derived under the assumption

$$\mathbb{E}[X_t^2 | X_{t-1}, X_{t-2}, \ldots] - \lambda_t(\theta_0)^2 =: v_t(\theta_0), \qquad a.s..$$

Under which minimal conditions on F the LHS exists?

Ferland et al. (2006)

If X_t is conditionally Poisson then $v_t = 0$ and the LHS exists under the conditions of Doukhan and W. (2008).

Moments for AR models with random coefficients and finite order

AutoRegressive models of infinite order AR(P) with random coefficients, $p \in \ltimes \cup \{\infty\}$

AR(p) model with random coefficients

Let $(A_j)_{j \geqslant 1}$, B be non-negative random variable. Consider the recurrence

$$X_t = \sum_{j=1}^{p} A_{t,j} X_{t-j} + B_t\,, \qquad t \in \mathbb{Z}\,, \qquad p \in \mathbb{N} \cup \{\infty\}\,,$$

where $((A_{t,j})_{j \geqslant 1}, B_t)_{t \in \mathbb{Z}}$ is an iid sequence with generic element $((A_j)_{j \geqslant 1}, B)$.

Review of the results in the litterature for $p < \infty$.

Stochastic Recurrence Equation SRE

The solution of the AR(p) model with random coefficients satisfies the SRE

$$\boldsymbol{X}_t = \boldsymbol{A}_t \boldsymbol{X}_{t-1} + \boldsymbol{B}_t\,, \qquad t \in \mathbb{Z}\,,$$

with the matrix notation

$$\boldsymbol{A}_t := \begin{pmatrix} A_{t,1} & A_{t,2} & \cdots & A_{t,p} \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix}, \qquad \boldsymbol{B}_t := \begin{pmatrix} B_t \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \boldsymbol{X}_t := \begin{pmatrix} X_t \\ \vdots \\ X_{t-p+1} \end{pmatrix}.$$

Kesten theory

Theorem (Kesten, 1973, Buraczewski et al. 2018)

Consider the convex function

$$h(\theta) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left[\|\boldsymbol{A}_n \cdots \boldsymbol{A}_1\|^{\theta}\right], \qquad \theta > 0.$$

If there exist $0 < \alpha < \alpha'$, *such that* $h(\alpha) = 0$, $h(\alpha') < \infty$ *and* $\mathbb{E}[B^{\alpha'}] < \infty$, *and under a further non-lattice assumption, there exists a stationary solution* $(\boldsymbol{X}_t)$ *to the SRE with generic element* $\boldsymbol{X}$ *satisfying*

$$\mathbb{P}(\|\boldsymbol{X}\| > x) \sim C x^{-\alpha}, \qquad x \to \infty.$$

Then $\mathbb{E}[X^{\theta}] < \infty$ *for* $\theta < \alpha$ *and* $\mathbb{E}[X^{\theta}] = \infty$ *otherwise.*

Explicit conditions

Moment of order 1

By linearity the sufficient condition $h(1) < 0$ is equivalent to

$$\sum_{j=1}^{p} \mathbb{E}[A_j] < 1\,.$$

Theorem (Nicholls and Quinn, 1982, Pham 1986)

Denote by ρ the spectral radius and $\otimes$ the tensor product. The sufficient condition of $h(2) < 0$ is equivalent to

$$\rho(\mathbb{E}[\boldsymbol{A} \otimes \boldsymbol{A}]) < 1\,.$$

Moments for AR models with random coefficients and infinite order

Smoothing transform

Consider the recurrence

$$X_t = \sum_{j=1}^{\infty} A_{t,j} X_{t-j} + B_t \,, \qquad t \in \mathbb{Z}\,.$$

Definition

The smoothing transform is the solution of the distributional equation

$$Y \overset{d}{=} \sum_{j=1}^{\infty} A_j Y^{(j)} + B,$$

where $(Y^{(j)})_{j\geqslant 1}$ are iid copies of Y, independent of $((A_j)_{j\geqslant 1}, B)$.

Theorem (Buraczewski et al., 2018)

Using the notation

$$\varphi_1(\theta) := \log \sum_{j=1}^{\infty} \mathbb{E}\big[A_j^{\theta}\big]\,, \qquad \theta > 0\,,$$

the smoothing transform exists if for some $\theta \in (0,1]$, $\mathbb{E}[B^{\theta}] < \infty$ *and* $\varphi_1(\theta) < 0$, *and then* $\mathbb{E}[Y^{\theta}] < \infty$.

Representation of the stationary solution

Consider the recurrence

$$X_t = \sum_{j=1}^{\infty} A_{t,j} X_{t-j} + B_t \,, \qquad t \in \mathbb{Z} \,.$$

Sums of products

A non-anticipative stationary solution (X_t) of the AR(∞) with generic element admitting the representation

$$\widetilde{X} = \sum_{0=t_0<t_1<\cdots<t_n,\ n\geqslant 0} \widetilde{A}_{t_0,t_1-t_0} \cdots \widetilde{A}_{t_{n-1},t_n-t_{n-1}} B_{-t_n} \,,$$

where $\widetilde{A}_{t,j} = A_{-t,j}$, $t \in \mathbb{Z}$, $j \geqslant 1$.

Representation of the stationary solution

Recall the sum of products

$$\widetilde{X} = \sum_{0=t_0<t_1<\cdots<t_n,\ n\geqslant 0} \widetilde{A}_{t_0,t_1-t_0} \cdots \widetilde{A}_{t_{n-1},t_n-t_{n-1}} B_{-t_n}\,.$$

Applying Minkowski's inequality we get for every $\theta \geqslant 1$,

$$\mathbb{E}[\widetilde{X}^{\theta}]^{1/\theta} \leqslant \sum_{0=t_0<t_1<\cdots<t_n,\ n\geqslant 0} \mathbb{E}[(\widetilde{A}_{t_0,t_1-t_0} \cdots \widetilde{A}_{t_{n-1},t_n-t_{n-1}} B_{-t_n})^{\theta}]^{1/\theta}\,.$$

Definition

We introduce the notation

$$\widetilde{\varphi}_1(\theta) := \log \sum_{j=1}^{\infty} \mathbb{E}\left[A_j^{\theta}\right]^{1/\theta}, \qquad \theta > 0\,.$$

First order condition of moments

By subadditivty we have

$$\begin{cases} \theta\widetilde{\varphi}_1(\theta) \leqslant \varphi_1(\theta), & 0 < \theta \leqslant 1, \\ \varphi_1(\theta) \leqslant \theta\widetilde{\varphi}_1(\theta), & \theta \geqslant 1, \end{cases}$$

and $\varphi_1(1) = \widetilde{\varphi}_1(1)$.

Theorem (Maillard and W., 2024)

Let (X_t) satisfies the AR(∞) model with $X_{-1} = X_{-2} = \cdots = 0$.

- *As $t \to \infty$, X_t converges in law to a (possibly infinite) limit $\widetilde{X}$. Moreover if*

1. *For some $0 < \theta \leqslant 1$*
   1.1 *$\varphi_1(\theta) < 0$ and $\mathbb{E}[B^\theta] < \infty$ then $\mathbb{E}[\widetilde{X}^\theta] < \infty$.*
   1.2 *$\widetilde{\varphi}_1(\theta) \geqslant 0$ or $\mathbb{E}[B^\theta] = \infty$ then $\mathbb{E}[\widetilde{X}^\theta] = \infty$.*
2. *For some $\theta \geqslant 1$*
   2.2 *$\widetilde{\varphi}_1(\theta) < 0$ and $\mathbb{E}[B^\theta] < \infty$ then $\mathbb{E}[\widetilde{X}^\theta] < \infty$.*
   2.1 *$\varphi_1(\theta) \geqslant 0$ or $\mathbb{E}[B^\theta] = \infty$ then $\mathbb{E}[\widetilde{X}^\theta] = \infty$.*

Second order conditions

Definition (Finite increasing integer-valued sequences)

Define

$$\mathscr{T} = \{\boldsymbol{t} = (t_0, \dots, t_n) : n \geqslant 0,\ 0 = t_0 < t_1 < \cdots < t_n\} \ ,$$

$\mathbf{0} = (0)$, $\boldsymbol{t} = (t_0, \dots, t_n) \in \mathscr{T}$, $n(\boldsymbol{t}) = n$ and $\widetilde{A}_{\boldsymbol{t}} = \widetilde{A}_{t_0, t_1 - t_0} \cdots \widetilde{A}_{t_{n-1}, t_n - t_{n-1}}$, with $\widetilde{A}_{\mathbf{0}} = 1$ by convention. We also denote $\widetilde{B}_{\boldsymbol{t}} := B_{-t_n}$.

With this notation, we have

$$\widetilde{X} = \sum_{\boldsymbol{t} \in \mathscr{T}} \widetilde{A}_{\boldsymbol{t}} \widetilde{B}_{\boldsymbol{t}},$$

and hence

$$\widetilde{X}^2 = \sum_{\boldsymbol{s}, \boldsymbol{t} \in \mathscr{T}} \widetilde{A}_{\boldsymbol{s}} \widetilde{B}_{\boldsymbol{s}} \widetilde{A}_{\boldsymbol{t}} \widetilde{B}_{\boldsymbol{t}}.$$

Definition (Closed and open pairs)

Denote by $\mathscr{C}$ and $\mathscr{O}$ the set of *closed* and *open* pairs

$$\mathscr{C} = \{(\boldsymbol{s}, \boldsymbol{t}) \in \mathscr{T} \times \mathscr{T} : n(\boldsymbol{s}) > 0,\ n(\boldsymbol{t}) > 0,\ s_{n(\boldsymbol{s})} = t_{n(\boldsymbol{t})},$$
$$s_i \neq t_j,\ 0 < i < n(\boldsymbol{s}),\ 0 < j < n(\boldsymbol{t})\}$$
$$\mathscr{O} = \{(\boldsymbol{s}, \boldsymbol{t}) \in \mathscr{T} \times \mathscr{T} : \forall 0 < i \leqslant n(\boldsymbol{s}),\ 0 < j \leqslant n(\boldsymbol{t}) : s_i \neq t_j\}.$$

Define the concatenation as

$$\boldsymbol{t}^1 \cdots \boldsymbol{t}^k := (t^1_0, \ldots, t^1_{n(\boldsymbol{t}^1)}, t^1_{n(\boldsymbol{t}^1)} + t^2_1, \ldots, t^1_{n(\boldsymbol{t}^1)} + t^2_{n(\boldsymbol{t}^2)}, \ldots, t^1_{n(\boldsymbol{t}^1)} + \cdots + t^k_{n(\boldsymbol{t}^k)}).$$

Decomposition of pairs

For every $(\boldsymbol{s}, \boldsymbol{t}) \in \mathscr{T} \times \mathscr{T}$ there exists $k \in \mathbb{N}_0$, $(\boldsymbol{s}^1, \boldsymbol{t}^1), \ldots, (\boldsymbol{s}^k, \boldsymbol{t}^k) \in \mathscr{C}$ and $(\boldsymbol{s}^o, \boldsymbol{t}^o) \in \mathscr{O}$, such that

$$(\boldsymbol{s}, \boldsymbol{t}) = (\boldsymbol{s}^1 \cdots \boldsymbol{s}^k \boldsymbol{s}^o, \boldsymbol{t}^1 \cdots \boldsymbol{t}^k \boldsymbol{t}^o).$$

Decomposition of pairs

Note that the trivial pair is open by definition: $(\mathbf{0}, \mathbf{0}) \in \mathscr{O}$.

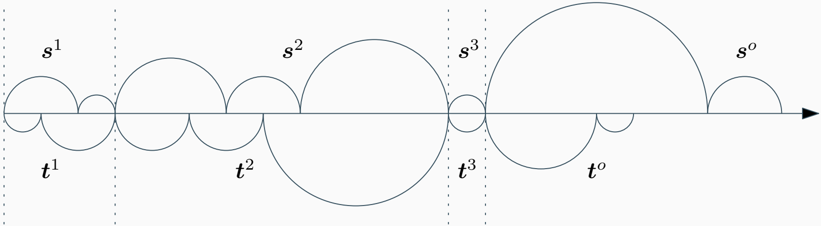

Figure 1: Schematic illustration of the decomposition of a pair $(\boldsymbol{s}, \boldsymbol{t}) \in \mathscr{T} \times \mathscr{T}$ into finitely many (here, three) closed pairs $(\boldsymbol{s}^1, \boldsymbol{t}^1)$, $(\boldsymbol{s}^2, \boldsymbol{t}^2)$, $(\boldsymbol{s}^3, \boldsymbol{t}^3)$ and an open pair $(\boldsymbol{s}^o, \boldsymbol{t}^o)$.

Second order test functions

Definition

$$\varphi_2(\theta) := \log \sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathscr{C}} \mathbb{E}[\widetilde{A}_{\boldsymbol{s}}^{\theta/2}\widetilde{A}_{\boldsymbol{t}}^{\theta/2}]\,, \qquad \theta > 0,$$

$$\widetilde{\varphi}_2(\theta) := \log \sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathscr{C}} \mathbb{E}[\widetilde{A}_{\boldsymbol{s}}^{\theta/2}\widetilde{A}_{\boldsymbol{t}}^{\theta/2}]^{2/\theta}\,, \qquad \theta > 0\,.$$

By subadditivty we have

$$\begin{cases} (\theta/2)\widetilde{\varphi}_2(\theta) \leqslant \varphi_2(\theta), & 0 < \theta \leqslant 2, \\ \varphi_2(\theta) \leqslant (\theta/2)\widetilde{\varphi}_2(\theta), & \theta \geqslant 2, \end{cases}$$

Considering the closed pairs $((0,i),(0,i))$, $i \geqslant 1$, we get

$$\varphi_2(\theta) \geqslant \varphi_1(\theta), \qquad \theta > 0.$$

Applying Cauchy-Schwarz's inquality and $\mathscr{C} \subset \mathscr{T}^2$ we obtain

$$\widetilde{\varphi}_2(\theta) \leqslant 2\widetilde{\varphi}_1(\theta), \qquad \theta > 0.$$

Second order condition of moments

Theorem (Maillard and W., 2024)

Let (X_t) satisfies the AR(∞) model with $X_{-1} = X_{-2} = \cdots = 0$.

1 *For some* $0 < \theta \leqslant 2$
   - 1.1 $\varphi_2(\theta) < 0$, $\varphi_2(\theta/2) < 0$ *and* $\mathbb{E}[B^\theta] < \infty$ *then* $\mathbb{E}[\widetilde{X}^\theta] < \infty$.
   - 1.2 $\widetilde{\varphi}_2(\theta) \geqslant 0$ *or* $\mathbb{E}[B^\theta] = \infty$ *then* $\mathbb{E}[\widetilde{X}^\theta] = \infty$.

2 *For some* $\theta \geqslant 2$
   - 2.2 $\widetilde{\varphi}_2(\theta) < 0$, $\widetilde{\varphi}_2(\theta/2) < 0$ *and* $\mathbb{E}[B^\theta] < \infty$ *then* $\mathbb{E}[\widetilde{X}^\theta] < \infty$.
   - 2.1 $\varphi_1(\theta) \geqslant 0$ *or* $\mathbb{E}[B^\theta] = \infty$ *then* $\mathbb{E}[\widetilde{X}^\theta] = \infty$.

Notice that $\varphi_2(2) = \widetilde{\varphi}_2(2)$ and

$$\varphi_2(2) < 0 \Leftrightarrow \sum_{(\boldsymbol{s},\boldsymbol{t}) \in \mathscr{C}} \mathbb{E}[\widetilde{A}_{\boldsymbol{s}} \widetilde{A}_{\boldsymbol{t}}] < 1$$

is almost a necessary and sufficient condition for the existence of second order moments

Proof hints

We use the decomposition

$$\widetilde{X}^2 = \sum_{k=0}^{\infty} \sum_{(\boldsymbol{s}^1,\boldsymbol{t}^1),\ldots,(\boldsymbol{s}^k,\boldsymbol{t}^k)\in\mathscr{C}} \sum_{(\boldsymbol{s}^o,\boldsymbol{t}^o)\in\mathscr{O}} \widetilde{A}_{\boldsymbol{s}^1\ldots\boldsymbol{s}^k\boldsymbol{s}^o} \widetilde{B}_{\boldsymbol{s}^1\ldots\boldsymbol{s}^k\boldsymbol{s}^o} \widetilde{A}_{\boldsymbol{t}^1\ldots\boldsymbol{t}^k\boldsymbol{t}^o} \widetilde{B}_{\boldsymbol{t}^1\ldots\boldsymbol{t}^k\boldsymbol{t}^o}.$$

The independence of $\widetilde{A}_{t,j}$ and $\widetilde{A}_{t',j'}$ for $t \neq t'$ yields

$$\mathbb{E}[\widetilde{A}_{\boldsymbol{s}^1\ldots\boldsymbol{s}^k\boldsymbol{s}^o} \widetilde{B}_{\boldsymbol{s}^1\ldots\boldsymbol{s}^k\boldsymbol{s}^o} \widetilde{A}_{\boldsymbol{t}^1\ldots\boldsymbol{t}^k\boldsymbol{t}^o} \widetilde{B}_{\boldsymbol{t}^1\ldots\boldsymbol{t}^k\boldsymbol{t}^o}] = \mathbb{E}[\widetilde{A}_{\boldsymbol{s}^1}\widetilde{A}_{\boldsymbol{t}^1}]\cdots\mathbb{E}[\widetilde{A}_{\boldsymbol{s}^k}\widetilde{A}_{\boldsymbol{t}^k}]\mathbb{E}[\widetilde{A}_{\boldsymbol{s}^o}\widetilde{B}_{\boldsymbol{s}^o}\widetilde{A}_{\boldsymbol{t}^o}\widetilde{B}_{\boldsymbol{t}^o}].$$

And thus

$$\mathbb{E}[X^2] = \left(\sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathscr{O}} \mathbb{E}[\widetilde{A}_{\boldsymbol{s}}\widetilde{B}_{\boldsymbol{s}}\widetilde{A}_{\boldsymbol{t}}\widetilde{B}_{\boldsymbol{t}}]\right) \sum_{k=0}^{\infty} \left(\sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathscr{C}} \mathbb{E}[\widetilde{A}_{\boldsymbol{s}}\widetilde{A}_{\boldsymbol{t}}]\right)^k.$$

Finally by definition of $\mathscr{O}$, the sets S and T are disjoint hence $\widetilde{A}_{\boldsymbol{s}}$ and $\widetilde{A}_{\boldsymbol{t}}$ are independent and

$$\begin{aligned}\sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathscr{O}} \mathbb{E}[\widetilde{A}_{\boldsymbol{s}}\widetilde{B}_{\boldsymbol{s}}\widetilde{A}_{\boldsymbol{t}}\widetilde{B}_{\boldsymbol{t}}] &= \mathbb{E}[B^2] + \sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathscr{O}\setminus(\mathbf{0},\mathbf{0})} \mathbb{E}[\widetilde{A}_{\boldsymbol{s}}]\mathbb{E}[\widetilde{A}_{\boldsymbol{t}}]\mathbb{E}[B]^2 \\ &\leqslant \mathbb{E}[B^2] + \sum_{\boldsymbol{s},\boldsymbol{t}\in\mathscr{T}} \mathbb{E}[\widetilde{A}_{\boldsymbol{s}}]\mathbb{E}[\widetilde{A}_{\boldsymbol{t}}]\mathbb{E}[B]^2 \\ &= \mathbb{E}[B^2] + \mathbb{E}[X]^2.\end{aligned}$$

Comparaison with Nicholls and Quinn's condition

Consider the AR(p) model with random coefficients or equivalently the SRE

$$X_t = \sum_{j=1}^{p} A_{t,j} X_{t-j} + B_t \,, \qquad t \in \mathbb{Z}\,,$$

$$\boldsymbol{X}_t = \boldsymbol{A}_t \boldsymbol{X}_{t-1} + \boldsymbol{B}_t \,, \qquad t \in \mathbb{Z}\,.$$

Theorem (Maillard and W. (2024))

Assume that $A_j > 0$ a.s. for every $1 \leqslant j \leqslant p$ then

$\left[\sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathscr{C}} \mathbb{E}[\widetilde{A}_{\boldsymbol{s}}\widetilde{A}_{\boldsymbol{t}}] < 1 \text{ and } \sum_{j=1}^{\infty} \mathbb{E}[A_j] < 1\right]$ *if and only if* $\rho(\mathbb{E}[\boldsymbol{A} \otimes \boldsymbol{A}]) < 1$.

Proof Hints

Check that $\rho(\mathbb{E}[\boldsymbol{A} \otimes \boldsymbol{A}]) < 1$ is necessary and combine with the sufficiency in Pham (1986).

Illustration on GARCH(1,1) volatility

Consider the equation satisfied by the volatility of the GARCH(1,1) model

$$X_t = 1 + \beta(1 + Z_t)X_{t-1}\,, \qquad t \in \mathbb{Z}\,,$$

where $(Z_t)_{t\in\mathbb{Z}}$ is an iid sequence of copies of a non-negative random variable Z. For every $\theta > 0$,

$$\mathbb{E}[X^\theta] < \infty \iff \log(\mathbb{E}[\beta^\theta (1 + Z)^\theta]) < 0.$$

The volatility of the GARCH(1,1) model also satisfies

$$X_t = \frac{1}{1-\beta} + \sum_{k\geqslant 0} \beta^{k+1} Z_{t-k} X_{t-1-k}\,, \qquad t \in \mathbb{Z}\,,$$

Setting $A_{t,j} = \beta^j Z_{t-j+1}$, $j \geqslant 1$, $t \in \mathbb{Z}$, we recognize an AR(∞) model with random coefficients.

Test functions for the GARCH(1,1) volatility

Test functions from the AR(∞) modeling

$$\varphi_1(\theta) = \log\left(\frac{\beta^\theta \mathbb{E}[Z^\theta]}{1-\beta^\theta}\right),$$

$$\tilde{\varphi}_1(\theta) = \log\left(\frac{\beta \mathbb{E}[Z^\theta]^{1/\theta}}{1-\beta}\right),$$

$$\varphi_2(\theta) = \begin{cases} \log\left(\dfrac{\beta^\theta \mathbb{E}[Z^\theta]}{1-\beta^\theta(1+2\mathbb{E}[Z^{\theta/2}])}\right), & \text{if } \beta^\theta(1+2\mathbb{E}[Z^{\theta/2}]) < 1 \\ +\infty, & \text{otherwise.} \end{cases}$$

$$\tilde{\varphi}_2(\theta) = \begin{cases} \log\left(\dfrac{\beta^2 \mathbb{E}[Z^\theta]^{2/\theta}}{1-\beta^2(1+2\mathbb{E}[Z^{\theta/2}]^{2/\theta})}\right), & \text{if } \beta^2(1+2\mathbb{E}[Z^{\theta/2}]^{2/\theta}) < 1 \\ +\infty, & \text{otherwise.} \end{cases}$$

Illustration

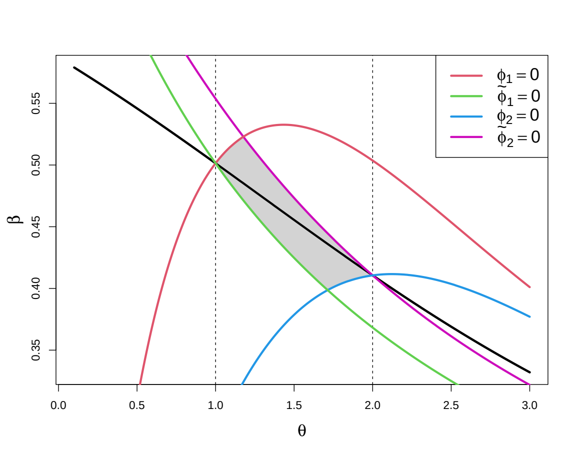

Figure 2: Illustration of our necessary and sufficient conditions of moments applied to the marginal solution with (Z_t) iid χ_1^2-distributed. Our approach is not conclusive in the grey region.

Conclusions and perspectives

Conclusions

- We obtain second order conditions for moments that are tractable and improve the first order conditions of moments when $\theta \approx 2$,
- We were not able to extend the approach to higher orders because the combinatorics gets more and more involved.

Perspectives

- Apply a domination argument to get a second order condition of moments on general chains with infinite memory,
- Apply this argument to causal expressions of standard processes and use the existence of moments to prove the asymptotic normality of Gaussian QMLE (Bardet and W., 2009) or PQMLE (Ahmad and Francq, 2016).

Thank you for your attention!