Clustering in large deviationss

with A. Chakrabarty

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- $X_0 + X_1 + \ldots + X_{n-1}$ is "too far from the mean" for a weak limit,
- $\max(X_0, X_1, \ldots, X_{n-1})$ is "too large" for a weak limit ...

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- If $E_0(n)$ occurs, how many of "nearby" $E_i(n)$ occur?

Infinite moving average processes

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- (X_n): stationary, ergodic, 0 mean, finite exponential moments

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- Does the overall sum stay above $n\varepsilon$?

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given $E_0(n,\varepsilon)$ has a non-degenerate weak limit,

it is a function of a certain overshoot process.

For fixed $K_-, K_+ \in \mathbb{Z}$,

$$P\left(\left(\sum_{j=-K_-}^{-1} \mathbf{1}(E_j(n,\varepsilon)), \sum_{j=1}^{K_+} \mathbf{1}(E_j(n,\varepsilon)) \right) \in \cdot \middle| E_0(n,\varepsilon) \right)$$

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The total size of the large deviation cluster converges weakly (after normalization) as $\varepsilon \to 0$.

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- Log-Laplace transform of the noise:

$$\varphi_Z(t) = \log\left(\int_{\mathbb{R}} e^{tz}\, F_Z(dz)\right),\ t \in \mathbb{R}.$$

- Exponentially tilted noise distribution:

$$G_\theta(dx) = e^{-\varphi_Z(\theta)+\theta x} F_Z(dx), \; x \in \mathbb{R}, \; \theta \in \mathbb{R}.$$

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  We assume $0 < \varepsilon/A < s_0$.

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- Conditioning on $E_0(n, \varepsilon)$ leads to exponential tilting for noise:

- $\{Z_j^u : j \in \mathbb{Z},\ u = + \text{ or } -\}$ independent random variables with

$$\begin{aligned}
Z_{-j}^- &\sim G_{\tau(\varepsilon)(A^+ - A_{j-1}^+)/A}\,, & j \geq 1\,,\\
Z_j^- &\sim G_{\tau(\varepsilon)(A^+ + A_j^-)/A}\,, & j \geq 0\,,\\
Z_{-j}^+ &\sim G_{\tau(\varepsilon)(A_{j-1}^+ + A^-)/A}\,, & j \geq 1\,,\\
Z_j^+ &\sim G_{\tau(\varepsilon)(A^- - A_j^-)/A}\,, & j \geq 0\,.
\end{aligned}$$

- Denote

$$U_n^- = \sum_{i=-\infty}^{\infty} a_i Z_{n-i}^-, \; n \in \mathbb{Z}, \quad U_n^+ = \sum_{i=-\infty}^{\infty} a_i Z_{n-i}^+, \; n \in \mathbb{Z}.$$

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- T^*: exponential random variable with parameter $\tau(\varepsilon)/A$, independent of the tilted Z's.

- Set

$$V_j(\varepsilon) = \begin{cases} \mathbf{1}\left(T^* \geq \sum_{i=0}^{j-1}(U_i^- - U_i^+)\right), & j \geq 1 \\ \mathbf{1}\left(T^* \geq \sum_{i=j}^{-1}(U_i^+ - U_i^-)\right), & j < 0. \end{cases}$$

Theorem 1

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Assume non-lattice case: $\left|\int_{\mathbb{R}} e^{itz}\,F_Z(dz)\right| < 1,\ t \neq 0$. Then

$$P\left(\left(\sum_{i=j}^{j+n-1} X_i - n\varepsilon,\, j \in \mathbb{Z}\right) \in \cdot \,\middle|\, E_0(n,\varepsilon)\right)$$
$$\Rightarrow \Bigg(T^* + \sum_{i=j}^{-1} U_i^- - \sum_{i=j}^{-1} U_i^+,\ j = \dots, -2, -1,$$
$$T^* - \sum_{i=0}^{j-1} U_i^- + \sum_{i=0}^{j-1} U_i^+,\ j = 0, 1, 2, \dots \Bigg).$$

By Theorem 1, for fixed $K_-, K_+ \in \mathbb{Z}$,

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\begin{aligned}
& P\left(\left(\sum_{j=-K_-}^{-1} \mathbf{1}(E_j(n,\varepsilon)), \sum_{j=1}^{K_+} \mathbf{1}(E_j(n,\varepsilon))\right) \in \cdot \,\middle|\, E_0(n,\varepsilon)\right) \\
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$$\Rightarrow P\left(\left(\sum_{j=-K_-}^{-1} V_j(\varepsilon), \sum_{j=1}^{K_+} V_j(\varepsilon)\right) \in \cdot\right).$$

- Total cluster size of large deviations:

$$\left(D_\varepsilon^-, D_\varepsilon^+\right) = \left(\sum_{j=-\infty}^{-1} V_j(\varepsilon), \sum_{j=1}^{\infty} V_j(\varepsilon)\right)$$

Theorem 2

$D_\varepsilon^-, D_\varepsilon^+ < \infty$ a.s. and, as $\varepsilon \to 0$,

$$\left(\varepsilon^2 D_\varepsilon^-, \varepsilon^2 D_\varepsilon^+\right) \Rightarrow \Bigg(A^2 \sigma_Z^2 \int_0^\infty \mathbf{1}\left(T_0 \geq \sqrt{2} B_t^- + t\right) dt, \\ A^2 \sigma_Z^2 \int_0^\infty \mathbf{1}\left(T_0 \geq \sqrt{2} B_t^+ + t\right) dt \Bigg)$$

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- $(B_t^\pm, t \geq 0)$ independent standard Brownian motions;
- T_0 independent standard exponential.

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- We assume that

$$a_n \sim n^{-\alpha}, \; n \to \infty, \; 1/2 < \alpha < 1.$$

- The moving average process

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is still ergodic, with finite exponential moments.

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- Covariances decay slower:

$$\sum_{n=0}^{\infty} |\mathrm{cov}(X_0, X_n)| < \infty, \ \ \text{short memory case,}$$

$$\mathrm{cov}(X_0, X_n) \sim Cn^{-(2\alpha-1)},\ n \to \infty, \ \ \text{long memory case.}$$

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$$-\log P(E_0(n, \varepsilon)) \sim \begin{cases} Cn^{2\alpha-1}\varepsilon^2 & \text{long memory case,} \\ C(\varepsilon)n & \text{short memory case.} \end{cases}$$

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- Medium deviations vs. large deviations.

- In the long memory case

$$P\big(E_j(n,\varepsilon)|E_0(n,\varepsilon)\big) \to 1,\ n \to \infty,\ \text{any } j \geq 1.$$

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- We consider

$$I_n(\varepsilon) = \inf\{j \geq 1 : E_j(n,\varepsilon) \text{ does not occur}\},\ n \geq 1.$$

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Conditionally on $E_0(n,\varepsilon)$, $I_n(\varepsilon) \to \infty$ as $n \to \infty$

- How fast does $I_n(\varepsilon)$ grow conditionally on $E_0(n, \varepsilon)$?

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- What is the limit of the conditional law of a properly normalized $I_n(\varepsilon)$?
- The Brownian motion with drift in the short memory case is replaced by a fractional Brownian motion with drift.

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$$H = 3/2 - \alpha \in (1/2,1):$$

determines the limiting fractional Brownian motion.

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  continuous zero mean Gaussian process, covariance function

$$E\left(B_H(s)B_H(t)\right) = \frac{1}{2}\left(s^{2H} + t^{2H} - |s - t|^{2H}\right),\ s, t \geq 0\,.$$

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- T_0: independent standard exponential random variable.

Define

$$\tau_\varepsilon = \inf \left\{ t \geq 0 : \; B_H(t) \leq (2C_\alpha)^{-1/2} \varepsilon t^{2H} - (C_\alpha/2)^{1/2} \sigma_Z^2 \varepsilon^{-1} T_0 \right\},$$

with

$$C_\alpha = \frac{B(1-\alpha, 2\alpha - 1)}{(1-\alpha)(3-2\alpha)}.$$

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Under technical assumptions on the noise,

$$P\left(n^{-\beta}I_n(\varepsilon)\in\cdot\,\Big|E_0(n,\varepsilon)\right)\Rightarrow P\left(\tau_\varepsilon\in\cdot\right),\ n\to\infty\,.$$

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- The distribution of τ_ε depends only on the variance of the noise.
- Difference between moderate deviations and large deviations.

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  Recall: cluster size $\approx \varepsilon^{-2}$ in the short memory case.

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- There is overshoot $X_0 + \cdots + X_{n-1} - n\varepsilon$ and the negative drift killing the overshoot.
- The overshoot is approximately exponential; mean proportional to $1/\varepsilon$, grows as $n^{2(1-\alpha)}$.
- Fluctuations around negative drift are now governed by fractional Brownian motion.

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