Quasi-infinitely divisible distributions

Alexander Lindner, Ulm University

Based on (joint) works with/of
I. Alexeev, D. Berger, A Khartov, M. Kutlu, L. Pan and K. Sato

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Infinitely divisible distributions

Definition: A probability distribution μ on $\mathbb{R}^d$ is infinitely divisible, if for every $n \in \mathbb{N}$ there exists a probability distribution μ_n on $\mathbb{R}^d$ such that

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i.e. if it has convolution roots of all orders.

Remark: There exists a one-to-one correspondence between infinitely divisible distributions and Lévy processes (in law).

Fourier transform/characteristic function of measure $\mu = \mathcal{L}(X)$

$$\widehat{\mu}(z) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i} z^T x} \, \mu(\mathrm{d}x) = \mathbb{E} \mathrm{e}^{\mathrm{i} z^T X}, \quad z \in \mathbb{R}^d$$

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Lévy-Khintchine formula:

A probability distribution μ on $\mathbb{R}^d$ is infinitely divisible if and only if there exist $A \in \mathbb{R}^{d\times d}$ positive semidefinifite, $\gamma \in \mathbb{R}^d$ and a Lévy measure ν on $\mathbb{R}^d$ (i.e. satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} (1 \wedge |x|^2) \, \nu(dx) < \infty$) such that

$$\widehat{\mu}(z) = \exp\left\{ i\gamma^T z - \frac{1}{2} z^T A z + \int_{\mathbb{R}^d} \left(e^{ix^T z} - 1 - ix^T z \mathbf{1}_{|x|\le 1} \right) \nu(dx) \right\}.$$

The triplet (A, ν, γ) is unique and called **characteristic triplet** of μ.

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Definition: A probability measure μ on $\mathbb{R}^d$ is **quasi-infinitely divisible (q.i.d.)**, if there exist two infinitely divisible distributions μ_1 and μ_2 on $\mathbb{R}^d$ such that

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Hence q.i.d. distributions appear in factorisation problems of infinitely divisible distributions.

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- Since then works by Berger (2019), Passeggeri (2020), Kadankova/Simon/Wang (2020), Khartov (2019, 2022), Alexeev and Khartov (2022, 2023), Kutlu (2021), Berger/Kutlu/Lindner (2022), Berger/Lindner (2022), and others.

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- Mathematical applications of quasi-infinitely divisible distributions.

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Suppose μ is q.i.d. and denote characteristic triplets of μ_1, μ_2 above by (A_1, ν_1, γ_1) and (A_2, ν_2, γ_2). Denote

$$\begin{aligned} A &:= A_1 - A_2 \in \mathbb{R}^{d\times d} \quad \text{(symmetric)}, \\ \gamma &:= \gamma_1 - \gamma_2, \\ \nu &:= \nu_1 - \nu_2 \end{aligned}$$

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By Lévy-Khintchine formula,

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- The symmetric matrix $A \in \mathbb{R}^{d\times d}$ must be positive semidefinite.
- Convolutions of q.i.d. distributions are q.i.d. (with characteristic triplets adding up).

Examples of quasi-infinitely divisible distributions

Theorem (Cuppens, 1969):
If a probability distribution μ has an atom of mass $> 1/2$, i.e. if $\exists\ c \in \mathbb{R}^d$ with $\mu(\{c\}) > 1/2$, then μ is q.i.d. with $A = 0$ and finite quasi-Lévy measure.

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Corollary: A Bernoulli distribution $b(1, p)$ is q.i.d. if and only if $p \neq 1/2$. Hence a binomial distribution $b(n, p)$ is q.i.d. if and only if $p \neq 1/2$.

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Corollary: The set of q.i.d. distributions is not closed with respect to weak convergence.

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Idea of proof: $\widehat{\mu}$ is periodic in all coordinates. Take Fourier series of $\log \widehat{\mu}$. The Wiener-Lévy theorem shows that the Fourier coefficients c_n are absolutely summable. The quasi-Lévy measure is then of the form $\sum c_n \delta_n$ (when $h = 1$).

Further classes of quasi-infinitely divisible distributions

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Observe: If μ is concentrated on $\mathbb{Z}^d$, then $\widehat{\mu}$ is 2π-periodic in all coordinates (and continuous), hence $\inf_{z\in\mathbb{R}^d} |\widehat{\mu}(z)| > 0$ is equivalent to $\widehat{\mu}(z) \neq 0$ for all $z \in \mathbb{R}^d$. Hence generalise result of L./Pan/Sato (2018) and Berger/L. (2022).

Theorem (Berger (2019), Berger and Kutlu (2022)):
Let $\mu = p\mu_d + (1-p)\mu_{ac}$ be a probability distribution on $\mathbb{R}$, where μ_d is discrete, μ_{ac} absolutely continuous and $p \in (0,1]$. Then the following are equivalent:

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In that case, the quasi-Levy measure ν of μ is of the form

$$\nu(\mathrm{d}x) = \nu_1(\mathrm{d}x) + \frac{m\mathrm{e}^{-|x|}}{|x|}\mathrm{sgn}(x)\,\mathrm{d}x,$$

where ν_1 is a finite signed measure and $m \in \mathbb{Z}$. The Gaussian variance $A = 0$.

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Idea of proof:

$$\mu = N(0, a_1) * \underbrace{(p_1 \delta_0 + p_2 N(0, a_2 - a_1) + \ldots + p_n N(0, a_n - a_1))}_{\text{q.i.d. by previous Theorem}}$$

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Idea of proof: Every probability distribution on $\mathbb{R}$ can be approximated by a lattice distribution with finite support and zero-free characteristic function.

Theorem (Kutlu, 2021):
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Idea of proof: When $d = 2$, take

$$\mu := \frac{1}{3}\delta_{(0,1)} + \frac{1}{3}\delta_{(1,0)} + \frac{1}{3}\delta_{(1,1)}.$$

An application of the Poincaré-Miranda theorem (a multi-variable generalisation of the intermediate value theorem) shows that μ cannot be approximated by probability distributions with zero-free characteristic functions.

Cramér–Wold devices

Recall:

- A sequence of random vectors $(X_n)_{n\in\mathbb{N}}$ converges weakly to a random vector X if and only if $a^T X_n$ converges weakly to $a^T X$ for every $a \in \mathbb{R}^d$ (classical Cramér–Wold device).

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- **Not true for $\alpha \in (0,1)$:** D. Marcus (1983) gives an example of a distribution $\mathcal{L}(X)$ that is not α-stable but such that $\mathcal{L}(a^T X)$ is α-stable for each $a \in \mathbb{R}^d$.

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- Giné and Hahn (1983) show that the example of Marcus cannot be infinitely divisible.
- Hence there are distributions $\mu = \mathcal{L}(X)$ that are not infinitely divisible but such that $\mathcal{L}(a^T X)$ is infinitely divisible $\forall\, a \in \mathbb{R}^d$.

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- Or an example by Ibragimov (1972) who constructs a quasi-infinitely divisible distribution with truly signed quasi-Lévy measure but such that $\mathcal{L}(a^T X)$ is infinitely divisible for all $a \in \mathbb{R}^d$.

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Idea of proof: If the characteristic function of X is zero-free, then $\mathcal{L}(X)$ is quasi-infinitely divisible with finite quasi-Lévy measure. Can then relate the quasi-Lévy measures of $\mathcal{L}(a^T X)$ to that of $\mathcal{L}(X)$.

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The example of D. Marcus (1983)

- Let $\alpha \in (0,1)$ and $d = 2$. Let $\beta > 0$ small enough. Then

  $$\varphi(z) = \mathrm{e}^{-|z|^\alpha}\,\mathrm{e}^{\mathrm{i}\beta|z|\cos(3\theta)}, \quad \text{where} \quad z = |z|(\cos(\theta), \sin(\theta)) \in \mathbb{R}^2$$

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- $\mathcal{L}(X)$ is quasi-infinitely divisible with quasi-Lévy measure

$$\nu(B) = \underbrace{c_1 \int_0^{2\pi} \int_0^\infty \mathbf{1}_B(r\mathrm{e}^{\mathrm{i}\theta}) \frac{\mathrm{d}r}{r^{1+\alpha}} \, \mathrm{d}\theta}_{\text{rot. sym.}\alpha\text{-stable}} + \underbrace{c_2 \int_0^{2\pi} \int_0^\infty \mathbf{1}_B(r\mathrm{e}^{\mathrm{i}\theta}) \frac{\mathrm{d}r}{r^{1+1}} \, (-\cos(3\theta)) \, \mathrm{d}\theta}_{\text{`signed'}1\text{-stable}}.$$

  with constants $c_1, c_2 > 0$ (Berger, Kutlu and L., in preparation).

Happy birthday and all the best to you, Paul!