Some ideas about modeling real situations using hypoelliptic SDE equations with or without boundary conditions

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The present talk is deeply inspired in the paper:

Level set and drift estimation from a reflected Brownian motion with drift. Cholaquidis et al. Stat. Sin. 31, No. 1, 29-51 (2021).

This work models the movement of a herd of elephants using an elliptic diffusion on the plane with Neumann boundary condition on a compact region $\Omega \subset \mathbb{R}^2$. The authors estimate several characteristics of such a diffusion as the Home Range and also the drift under which the movement takes place.

However, this way of modeling implies that the trajectories of the animals are those of an elliptic diffusion that are non-differentiable at any point. The need to address this problem have motivated in the following paper

Employing stochastic differential equations to model wildlife motion. Brillinger et al Bull. Braz. Math. Soc. (N.S.) 33, No. 3, 385-408 (2002).

to propose a hypoelliptic model.

Let's talk to these authors. "*The movement in Newtonian dynamics may be described by a potential function* $V(x,t)$ *the equations of motion take the form*

$$\begin{aligned} d\mathbf{X}(t) &= \mathbf{V}(t)dt \\ d\mathbf{V}(t) &= -(c\mathbf{V}(t) + \beta\nabla_{\mathbf{x}} V(\mathbf{X}(t),t))dt + \sigma d\mathbf{W}(t). \end{aligned} \quad (1)$$

Where $\mathbf{X}(t) = (x_1(t), x_2(t))$ *is the particle's location at time* t, $\mathbf{V}$ *is the particle's velocity and* $\mathbf{W}$ *is a white noise*".
The difficulty now lies in the fact that the particle motion takes place in a compact region of $\mathbb{R}^2$ and we must then implement a model with total or partial reflection at the boundary.

Let's consider $V(\mathbf{x}, t) = V(\mathbf{x})$. Our problem has also a PDE formulation. Indeed the Heat equation associated to this process is the following

$$\partial_t u = \frac{\sigma^2}{2}\Delta_{\mathbf{v}} u + \mathbf{v}.\nabla_{\mathbf{x}} u - (c\mathbf{v} + \beta\nabla_{\mathbf{x}} V(\mathbf{x})).\nabla_{\mathbf{v}} u.$$

Now we need to restraint the movement and assuming a boundary condition, we point out that this condition is related only to the position the derivative has not constraints.

Then the domain in which the particle moves is the cylinder $\Omega \times \mathbb{R}^2$. Hence the function $u(t, \mathbf{x}, \mathbf{v})$ must satisfy some boundary conditions in the set $\partial\Omega \times \mathbb{R}^2$.

It will be necessary to consider some sets for defining our boundary condition. Let's denote $\mathbf{n}(\mathbf{x})$ the normal vector to $\partial\Omega$ in $\mathbf{x}$, pointing outwards to Ω. Let's introduce

$$\Sigma_{+}^{\mathbf{x}} = \{(\mathbf{x}, \mathbf{v}) \in \partial\Omega \times \mathbb{R}^2 : \mathbf{v}.\mathbf{n}(\mathbf{x}) > 0, \},$$

$$\Sigma_{0}^{\mathbf{x}} = \{(\mathbf{x}, \mathbf{v}) \in \partial\Omega \times \mathbb{R}^2 : \mathbf{v}.\mathbf{n}(\mathbf{x}) = 0, \},$$

$$\Sigma_{-}^{\mathbf{x}} = \{(\mathbf{x}, \mathbf{v}) \in \partial\Omega \times \mathbb{R}^2 : \mathbf{v}.\mathbf{n}(\mathbf{x}) < 0, \},$$

that we name respectively outgoing, grazing and ingoing sets.

The especular reflection will be

$$u(t,\mathbf{x},\mathbf{v}')|_{\mathbf{x}\in\partial\Omega,\,\mathbf{v}'\in\Sigma_+} = u(t,\mathbf{x},\mathbf{v}' - 2(\mathbf{v}'.\mathbf{n}(\mathbf{x}))\mathbf{n}(\mathbf{x})). \qquad (2)$$

Also the reverse movement is modeled by

$$u(t,\mathbf{x},\mathbf{v}')|_{\mathbf{x}\in\partial\Omega,\,\mathbf{v}'\in\Sigma_+} = u(t,\mathbf{x},-\mathbf{v}'). \qquad (3)$$

Other reflection conditions can be also considered.

Below we show a figure that schematizes the process

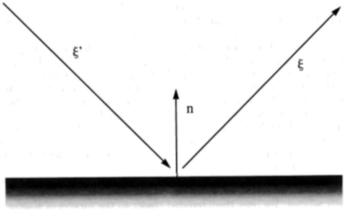

Figure: The reflection of a particle

The questions to study are

- How one can modify the system (1) to incorporate the boundary condition?
- As the solution need to be a Markov process, the steps for having the properties of ergodicity, exponential mixing and invariant measure could be implemented?
- If we observe one trajectory. It would be possible to estimate the Home Range, the boundary of the set where the movement takes place, the drift and the invariant density?

Let us recall some properties of the model without boundary conditions.
The system is

$$\begin{aligned} d\mathbf{X}(t) &= \mathbf{V}(t)dt \\ d\mathbf{V}(t) &= -(c\mathbf{V}(t) + \beta\nabla_{\mathbf{x}} V(\mathbf{X}(t))dt + \sigma d\mathbf{W}(t). \end{aligned} \tag{4}$$

In the seminal paper "*Large and moderate deviations and exponential convergence for stochastic damping Hamiltonian systems. Stochastic Process. Appl. 91, no. 2, 2052-38 (2001).*"

Liming Wu has proved a number of properties that satisfy these and more general models.

- 1. The semigroup is strong Feller.
  2. The density of transition is strictly positive
- Existence of a invariant measure μ related with the Hamiltonian function $H(x_1, x_2) = \frac{|x_2|^2}{2} + V(x_1)$ ($\sigma = 1$, $\beta = 1$ and $c = 1$)

$$d\mu(x_1, x_2) = \frac{e^{-2(\frac{|x_2|^2}{2} + V(x_1))}}{\int_{\mathbb{R}^2} e^{-2(\frac{|x_2|^2}{2} + V(x_1))} dx_1 dx_2} dx_1 dx_2.$$

- Existence of a Liapunov function $\Psi(\cdot)$.
- By using the Meyn-Tweedie theory it holds that the process is β-mixing and we have

$$||P_t(z, \cdot) - \mu||_{TV} \leq D'' \Psi(z) \rho^t, \text{ for } \rho < 1.$$

Let us return to our problem. To make more easy the presentation I only consider that the movement takes place in $[-1,1]$.
To be specific we have $\Omega=(-1,1)$ and $\partial\Omega=\{-1,1\}$. As before we decompose the set $\Sigma=\{-1,1\}\times\mathbb{R}$, which is the boundary of the cylinder $(-1,1)\times\mathbb{R}$, in the outgoing, grazing and ingoing boundary sets, introduced previously

$$\begin{aligned}\Sigma_+ &= \{(1,v)|v>0\}\cup\{(-1,v)|v<0\},\\ \Sigma_0 &= \{(1,0),\,(-1,0)\}\\ \Sigma_- &= \{(1,v)|v<0\}\cup\{(-1,v)|v>0\}.\end{aligned}$$

Moreover, the Fokker-Planck equation with specular reflection at the boundary for a function $u(t,x,v)$ writes as

$$\begin{aligned} \partial_t u &= \frac{\sigma^2}{2}\partial_{vv} u + v\partial_x u - cv\partial_v u - \beta\partial_x V(x)\partial_v u \quad (5) \\ u(t,x,v)|_{\Sigma_-} &= u(t,x,-v) \text{ for } x=1 \text{ and } x=-1, \end{aligned}$$

as before for simplicity in the notation and w.l.o.g. we take $\sigma^2 = 1$, $c = 1$ and $\beta = 1$.

We will follow the procedure developed in the lecture notes: *Boundary conditions and subelliptic estimates for geometric Kramers-Fokker-Planck operators on manifolds with boundaries. F Nier, volume 252. American Mathematical Society, (2018).* However, our approach will be more probabilistic.

Considering as Wu the following system without restriction. We know the existence of a Markov process solution of the system of SDE

$$\begin{aligned} d\mathbf{X}(t) &= \mathbf{V}(t)dt \\ d\mathbf{V}(t) &= -(\mathbf{V}(t) + \partial_x V(\mathbf{X}(t))dt + dW(t). \end{aligned} \tag{6}$$

This system has a invariant measure and we can define following random variable

$$\begin{aligned} &\tau(z) = \tau(x, v) \\ &= \inf_{t \geq 0}\{(\mathbf{X}(t), \mathbf{V}(t)) \in \Sigma_+; (\mathbf{X}(0), \mathbf{V}(0)) = (x, v) \in (-1, 1)\times\mathbb{R}\cup\Sigma_-\}, \end{aligned}$$

that results a stopping time, the process hit the boundary of all bounded set.

To incorporate the boundary condition defining the specular reflection, the system (6) must be completed introducing a jump process J at times τ when $Z_{\tau^-} := (\mathbf{X}_{\tau^-}, \mathbf{V}_{\tau^-}) \in \Sigma_+$. In the specular reflexion the jump process is deterministic and it holds

$$J(Z_{\tau_j^-}) := J(\mathbf{X}_{\tau^-}, \mathbf{V}_{\tau^-}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \mathbf{X}_{\tau^-} \\ \mathbf{V}_{\tau^-} \end{bmatrix} = \begin{bmatrix} \mathbf{X}_{\tau^-} \\ -\mathbf{V}_{\tau^-} \end{bmatrix}.$$

This implies

$$\begin{aligned} (-1, -|v|) &\rightarrow (-1, |v|) \\ (1, |v|) &\rightarrow (1, -|v|) \text{ for } v \in \mathbb{R}. \end{aligned}$$

From the stopping time, the process starts afresh. The above definition can be expressed in the following condensed form. Let define $Z(t) = (\mathbf{X}(t), \mathbf{V}(t))$, the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ and the drift $b(z) = b(x, v) = \begin{bmatrix} 0 \\ -\partial_x V(x)) \end{bmatrix}$. Thus, we can formally write our diffusion with jumps as

$$d\tilde{Z}(t) = A\tilde{Z}(t)dt + b(\tilde{Z}(t))dt + \begin{bmatrix} 0 \\ dW(t) \end{bmatrix} + dJ(t).$$

Where we define

$$J(t) = \begin{bmatrix} 0 \\ -2\sum_{s\leq t} \mathbf{V}(s^-)\mathbf{1}_{\{-1,1\}}(\mathbf{X}(s)) \end{bmatrix}$$

The process can be built recursively as follows. Moreover, it holds that the resulting process has cadlag trajectories and it is defined as follows

$$\tilde{Z}(t) = z + \int_0^t (A\tilde{Z}(s) + b(\tilde{Z}(s))ds + \begin{bmatrix} 0 \\ W(t) \end{bmatrix} = Z(t)$$

$0 \leq t < \tau(z) := \tau_1 \ z = \begin{bmatrix} x \\ v \end{bmatrix} \in (-1,1) \times \mathbb{R} \cup \Sigma_-$.
In this form for $j = 1, 2 \ldots$ we have

$$\tilde{Z}(t) = J(\tilde{Z}_{\tau_j^-}) + \int_{\tau_j}^t (A\tilde{Z}(s))dt + b(\tilde{Z}(s))ds + \begin{bmatrix} 0 \\ (W(t) - W(\tau_j)) \end{bmatrix}$$

for $\tau_j \leq t < \tau_{j+1}$. where $\tau_{j+1} = \inf_{t \geq \tau_j} \{\tilde{Z}(t) \in \Sigma_+\}$.

This process is markovian and has cadlag trajectories that remain inside the set $[-1,1]\times\mathbb{R}$.
Furthermore, the x-coordinate of the process is a.s. a continuous function and the v-coordinate has a jump at each τ_j.

Now we will study the semigroup associated to this process. For a continuous and bounded function $f\in C([-1,1]\times\mathbb{R})$ (also for a measurable and bounded function) we define

$$P_t^{\tilde{Z}}(f)(z)=\mathbb{E}^z[f(\tilde{Z}(t))] \text{ for } z\in(0,1)\times\mathbb{R}\cup\Sigma_- .$$

On the space $C_b([-1,1]\times\mathbb{R})$ this is a collection of operators that each takes a continuous function and transforms it in a bounded function, being necessary to determine its range. In the following we will explore these things.

Let's introduce $u(t,z) = P_t^{\tilde{Z}}(f)(z)$, we will determine its behavior at the boundary. Consider for instance, a sequence $z_n \to z = (1, v)$ with $v > 0$ a point of Σ_+. If we define

$$\tau_{1,n} = \inf_{t \geq 0}\{(\mathbf{X}(t), \mathbf{V}(t)) \in \Sigma_+ ; \, (\mathbf{X}(0), \mathbf{V}(0)) \\ = z_n = (x_n, v_n) \in (-1,1) \times \mathbb{R} \cup \Sigma_-\},$$

then by continuity we have $\tau_{1,n} \to 0$. As a consequence for $t > 0$ it holds $\mathbf{1}_{(\omega:\, \tau_{1,n}(\omega) \leq t)} \to 1$. We readily get

$$u(t, z_n) = \mathbb{E}^{z_n}[f(Z(t))\mathbf{1}_{(0 \leq t < \tau_{1,n})}] + \mathbb{E}^{z_n}[f(Z(t))\mathbf{1}_{(\tau_{1,n} \leq t)}] \\ \to u(t, J(z)) \neq u(x, z),$$

because $\mathbf{1}_{(0 \leq t < \tau_{1,n})} \to 0$.

We study now the case when $z \in (-1,1) \times \mathbb{R} \cup \Sigma_-$ and let's consider $z_n \to z$. Proceeding in the same way as before we obtain

$$u(t, z_n) \to u(t, z).$$

These two facts show the jumps in the v variable of the function $u(t, z)$ when it approaches to the boundary set Σ_+.

For computing the infinitesimal generator let f be a continuous function, having a continuous derivative with respect to the variable x and two continuos derivatives with respect to v. We say that f belongs to $C^{1,2}((-1,1), \times\mathbb{R})$.

By a precise use of the Itô's lemma and taking into account the jumps we get

$$\lim_{h\to 0}\frac{P_t^{\tilde{Z}}(f)(z)-f(z)}{h}=\frac{1}{2}\partial_{vv}f(z)+v\partial_x f-v\partial_v f-\partial_x V(x)\partial_v f$$
$$:=-\mathcal{L}(f),\ \text{ for } z\in(-1,1)\times\mathbb{R}\cup\Sigma_-. \quad (7)$$

The theory of semigroups allows us to conclude, for this f, that $u(t,z) = P_t^{\tilde{Z}}(f)(z)$ is the solution of the following problem

$$\begin{aligned} \partial_t u(t,z) &= -\mathcal{L}u(t,z) \\ u(0,z) &= f(z) \\ u(t,z)|_{\Sigma_-} &= u(t,J(z)) \end{aligned} \tag{8}$$

That can be also expressed by $u(t,z) = e^{-t\mathcal{L}}f(z)$
and $\lim\limits_{z \to z_0 \in \Sigma_+} = u(t,J(z_0))$.

We can consider also the formal adjoint of $\mathcal{L}$ that we will denote by $\mathcal{L}^*$. We get readily

$$-\mathcal{L}^*\rho = \frac{1}{2}\partial_{vv}\rho - v\partial_x\rho + \partial_v(v\rho) + \partial_x V(x)\partial_v\rho.$$

It remains to study the extension of these semigroups and theirs generators for a large class of functions.

For a function u defined in $[-1,1]\times\mathbb{R}$ we define the trace operator $\gamma : u \to \gamma u(x,\cdot)$, $x\in\{-1,1\}$. That is $\gamma u : \{-1,1\}\times\mathbb{R}\to\mathbb{R}$ with $\gamma u(-1,v) = u(-1,v)$ and $\gamma u(1,v) = u(1,v)$ when these values can be defined.
Moreover, we can introduce the even and odd parts with respect to the variable v

$$\gamma_{even} u(x,v) = \frac{\gamma u(x,v) + \gamma u(x,-v)}{2},$$

$$\gamma_{odd} u(x,v) = \frac{\gamma u(x,v) - \gamma u(x,-v)}{2}.$$

The above definitions allows writing the adjoint problem as

$$\begin{aligned} \partial_t \rho(t,z) &= -\mathcal{L}^* \rho(t,z) \\ \rho(0,z) &= \rho_0(z) \\ \gamma_{odd}\rho(t,z) &= 0. \end{aligned} \tag{9}$$

There is a fact that simplifies the searching of a domain for both the semigroup and its generator.

It is the existence of an invariant measure. Let's introduce

$$p(x,v) = \frac{e^{-2(\frac{|v|^2}{2}+V(x))}}{\int_{\mathbb{R}}\int_{-1}^{1} e^{-2(\frac{|v|^2}{2}+V(x))}dxdv} \text{ for } (x,v) \in [-1,1]\times\mathbb{R}.$$

A direct computation shows that $\mathcal{L}^* p(x,v) = 0$ and also furthermore because it is an even function it satisfies the boundary condition (9). Hence we have $P_t^*(p)(z) = p(z)$. This function gives us the possibility for defining two isomorphic semigroups acting in $\mathbb{L}^2([-1,1]\times\mathbb{R}, dxdv)$. Thus let's define the isometry

$$\mathbf{J} : \mathbb{L}^2([-1,1]\times\mathbb{R},\ p(x,v)dxdv) \to \mathbb{L}^2([-1,1]\times\mathbb{R},\ dxdv),$$

defined as $\mathbf{J}(f) = p^{\frac{1}{2}} f$.

Introducing the following semigroups

$$\tilde{P}_t(f)(z) = \mathbf{J}P_t(\mathbf{J}^{-1}f)(x,v),\ f \in \mathbb{L}^2([-1,1]\times\mathbb{R}, dxdv),$$

and its adjoint $\tilde{P}_t^* g = \mathbf{J}^{-1}P_t^*(\mathbf{J}g)$. It holds that

$$||\tilde{P}_t||_{\mathbb{L}^2([-1,1]\times\mathbb{R},\, dxdv)} = 1, \quad ||\tilde{P}_t^*||_{\mathbb{L}^2([-1,1]\times\mathbb{R},\, dxdv)} = 1.$$

By denoting $-P_{+,(-1,1)}(V)$ the infinitesimal generator of $\tilde{P}_t^*$ we obtain from a simple computation for f smooth enough

$$-P_{+,(-1,1)}(V) = \frac{1}{2}\partial_{vv} f(z) + \frac{1}{2}(1-v^2)f(z) - v\partial_x f(z) + \partial_x V(x)\partial_v f(z),$$

and the equality holds in $\mathbb{L}^2([-1,1]\times\mathbb{R}, dxdv)$ if the respective derivatives exist in this space. Our boundary value problems write now

$$\begin{array}{cc} \partial_t\tilde{\rho} = -P_{+,(-1,1)}(V)\tilde{\rho} & \partial_t\tilde{u} = -P_{-,(-1,1)}(V)\tilde{u} \\ \tilde{\rho}(0,z) = \tilde{\rho}_0(z) & \tilde{u}(0,z) = f(z) \\ \gamma_{odd}\tilde{\rho}(t,x,v) = 0 & \gamma_{odd}\tilde{u}(t,x,v) = 0 \end{array}$$

Using the convergence of the resolvents we can prove that the set of functions $f \in \mathbb{L}^2([-1,1] \times \mathbb{R}, dxdv)$ having two derivatives with respect to v and one derivative with respect to x both in $\mathbb{L}^2([-1,1] \times \mathbb{R}, dxdv)$, is contained in the domain of two generators. The only thing left to consider is the boundary condition.

We introduce the quantum harmonic oscillator operator $\mathcal{O} = \frac{1}{2}(-\partial_{vv} + v^2)$. The following Sobolev type spaces are useful

$$\mathcal{H}^s = \{u \in \mathcal{S}'(\mathbb{R}) = (\frac{1}{2} + \mathcal{O})^{\frac{s}{2}} u \in \mathbb{L}^2(\mathbb{R})\}.$$

For each $x \in [-1, 1]$ we assume the function $u(x, \cdot) \in \mathcal{H}^1$ and we can define the Hilbertian norm

$$||u||_{\mathbb{L}^2([-1,1],\mathcal{H}^1)}$$

$$= \frac{1}{2}\int_{\mathbb{R}}\int_{-1}^{1} |\partial_v u|^2 dvdx + \int_{\mathbb{R}}\int_{-1}^{1} |vu|^2 dvdx + \int_{\mathbb{R}}\int_{-1}^{1} |u|^2 dvdx,$$

and also

$$||u||_{\mathbb{L}^2([-1,1],\mathcal{H}^2)} = \int_{-1}^{1}\int_{\mathbb{R}} |(\frac{1}{2} + \mathcal{O})u|^2 dvdx.$$

We need to introduce a space defined on the boundary

$\mathbb{L}^2(\partial Q, |v| dS(x) dv)$

$$= \{f : \{-1,1\} \times \mathbb{R} \to \mathbb{C} : \int_{\mathbb{R}} |v|(|f(-1,v)|^2 + |f(1,v)|^2) dv < \infty\},$$

on the boundary we choose $dS(x) = \delta_{-1}(x) + \delta_1(x)$. We denote

$$||f||^2_{\mathbb{L}^2(\partial Q, |v| dS(x) dv)} = \int_{\mathbb{R}} |v|(|f(-1,v)|^2 + |f(1,v)|^2) dv.$$

An integration by parts formula gives the following inequality

$$||\gamma u||_{\mathbb{L}^2(\partial Q, |v| dS(x) dv)} \leq \mathbf{C} ||u||_{\mathbb{L}^2([-1,1], \mathcal{H}^1)}.$$

This relation shows the continuity of the operator trace $\gamma : \mathbb{L}^2([-1,1],\mathcal{H}^1) \to \mathbb{L}^2(\partial Q, |v| dS(x) dv)$. Moreover, another two inequalities hold

$$||\gamma_{odd} u||_{\mathbb{L}^2(\partial Q, |v| dS(x) dv)} \leq \mathbf{C} ||u||_{\mathbb{L}^2([-1,1],\mathcal{H}^1)},$$

and the same for γ_{even}.

Now we consider that the initial condition satisfies

$$\tilde{\rho}_0 \in \mathbb{L}^2([-1,1],\mathcal{H}^1) \cap \mathbb{L}^2([-1,1],\mathcal{H}^2).$$

We have seem that this function belongs to the $D(P_{+,(-1,1)}(V))$. Thus

$$\tilde{\rho}(t,x,v) = e^{-t(P_{+,(-1,1)}(V))}\tilde{\rho}_0,$$

remains into the above set of functions. Moreover, if for a smooth enough sequence we have $\tilde{\rho}_{0n} \to \rho_0$ in $\mathbb{L}^2([-1,1]\times\mathbb{R}, dxdv)$ and $-P_{+,(-1,1)}(V)\tilde{\rho}_{0n} \to -P_{+,(-1,1)}(V)\rho_0$ being this last convergence also in $\mathbb{L}^2([-1,1]\times\mathbb{R}, dxdv)$.

Then

$$\tilde{\rho}_n(t,x,v) = e^{-t(P_{+,(-1,1)}(V))}\tilde{\rho}_{0n} \to \tilde{\rho}_(t,x,v) = e^{-t(P_{+,(-1,1)}(V))}\tilde{\rho}_0.$$

For each $t > 0$ this two functions belong to the above set of functions and the convergence also takes place in $\mathbb{L}^2([-1,1],\mathcal{H}^1)$. Using the continuity of the trace we obtain

$$\gamma\tilde{\rho}_n(t,x,v) \to \gamma\tilde{\rho}(t,x,v),$$

then

$$0 = \gamma_{odd}\tilde{\rho}_n(t,x,v) \to \gamma_{odd}\tilde{\rho}(t,x,v)$$

verifying also the boundary condition.

In the following we will study the first eigenvalue associated to the two operators. Let's consider the function $p^{\frac{1}{2}}$ we have

$$\tilde{P}_t(p^{\frac{1}{2}}) = p^{\frac{1}{2}} P_t(p^{-\frac{1}{2}} p^{\frac{1}{2}}) = p^{\frac{1}{2}} P_t(\mathbf{1}) = p^{\frac{1}{2}}.$$

Moreover,

$$\tilde{P}_t^*(p^{\frac{1}{2}}) = p^{-\frac{1}{2}} P_t^*(p^{\frac{1}{2}} p^{\frac{1}{2}}) = p^{-\frac{1}{2}} P_t^*(p) = p^{-\frac{1}{2}} p = p^{\frac{1}{2}}.$$

These two equalities show that 0 is an eigenvalue for the two generators. Remark that in the second line of equations we have used that $\mathcal{L}^*(p) = 0$.

If we denote Π_0 the projection in $\mathbb{L}^2([-1.1], \times\mathbb{R}, dxdv)$ over the space generated by $p^{\frac{1}{2}}$, in the book of Nier (corollary 8.2) (and as a consequence of a deep study of the resolvents) it is proved that there exists a $\tau > 0$ and a constant $\mathbf{C}_1$ such that

$$||\tilde{P}_t f - < f, p^{\frac{1}{2}} > p^{\frac{1}{2}}||_{\mathbb{L}^2([-1.1],\times\mathbb{R},dxdv)} \leq \mathbf{C}_1\, e^{-\tau t}.$$

This result can be translated to the space $\mathbb{L}^2([-1.1], \times\mathbb{R}, p(x,v)dxdv) := \mathbb{L}^2(p)$. Indeed

$$||\tilde{P}_t f - < f, p^{\frac{1}{2}} > p^{\frac{1}{2}}||^2_{\mathbb{L}^2([-1.1],\times\mathbb{R},dxdv)} = ||P_t(p^{-\frac{1}{2}} f) - < p^{-\frac{1}{2}} f, 1 >_{\mathbb{L}^2(p)} ||^2_{\mathbb{L}^2(p)}.$$

Then setting $g = p^{-\frac{1}{2}} f$ we get

$$||P_t(g) - < g, 1 >_{\mathbb{L}^2(p)} ||_{\mathbb{L}^2(p)} \leq \mathbf{C}_1 \, e^{-\tau t}, \text{ for all } g \in \mathbb{L}^2(p).$$

These facts exhibit the return to the equilibrium to the only invariant measure $d\mu(x, v) = pdxdv$.

Let's point out that recently two teens of french researchers have studied problems related with those that we have exposed The first one is

Lelièvre Tony, Ramil Mouad, Reygner Julien. A probabilistic study ot the kinetic Fokker-Planck equation in cylindrical domains. J. Evol. Equ. 22:38 (2022). For absorbing boundary conditions.

and the second is

Fonte Sanchez Claudia, Pierre Grabriel, Stephane Mischler. On the Krein-Rutman theorem and beyond. arXiv:2304.01799 January (2024).

For more general boundary conditions and models that the one considered here.