Phantom distributions for non-stationary time series as an averaging operation

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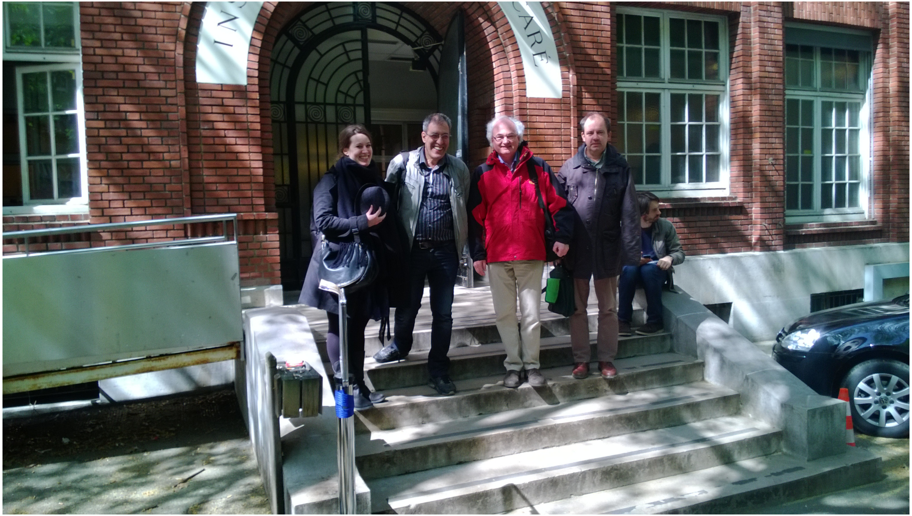
Cooking mathematics and steaks with the PI (2007–2024)

IV Nagaev's Lecture

Chaire Internationale 2013-2014, Labex MME-DII



A cook and a great companion

Cooking steaks

Enjoying friendship

The talk

Phantom distributions for non-stationary time series as an averaging operation

Joint work with Paul Doukhan

Phantom distribution functions

- The notion of a phantom distribution function was introduced by O'Brien, AoP(1987).
- Let $\{X_j\}$ be a stationary sequence with partial maxima

$$M_n = \max_{1 \leqslant j \leqslant n} X_j$$

  and the marginal distribution function $F(x) = \mathbb{P}(X_1 \leqslant x)$.
- A stationary sequence $\{X_n\}$ is said to admit a phantom distribution function G if

$$\sup_{u \in \mathbb{R}} \left| \mathbb{P}(M_n \leqslant u) - G^n(u) \right| \to 0, \text{ as } n \to \infty. \qquad (1)$$

- It is obvious that G is not uniquely determined for only the behavior of G at its right end $G_* = \sup\{x\,;\ G(x) < 1\}$ is of importance.
- When (1) is satisfied with $G(x) = F^\theta(x)$, for some $\theta \in (0,1]$, then we say that $\{X_j\}$ has the extremal index θ in the sense of Leadbetter, Z.Wahr(1983).

Phantom distribution functions are quite common

- Doukhan, J. and Lang, Extr(2015) showed that phantom distribution functions (in fact: continuous phantom distribution functions) exist in a large class of stationary sequences, including some non-ergodic ones.
- J., Mikosch, Rodionov and Soja-Kukieła (2022+) exhibited examples of stationary sequences with continuous phantom distribution functions but without the extremal index.
- J., Rodionov and Soja-Kukieła, Bern(2021) extended the notion of a phantom distribution function to stationary random fields, where interesting phenomena occur.
- In all examples given above we deal with stationarity, that seems to be a natural environment for the notion of a phantom distribution function

Phantom distribution function in the Metropolis algorithm

- Let $\{Z_j\}$ is an i.i.d. sequence with the marginal distribution function H given by the proposal density h, symmetric about 0.
- Let $\{U_j\}$ be an i.i.d. sequence distributed uniformly on $[0, 1]$, independent of $\{Z_j\}$.
- Let $f(x)$ be the target probability density.
- We consider the random walk Metropolis algorithm given by the recursive equation

$$X_{j+1} = X_j + Z_{j+1} \mathbb{1}_{\{U_{j+1} \leqslant \psi(X_j, X_j + Z_{j+1})\}},$$

where $\psi(x, y)$ is defined as

$$\psi(x, y) = \begin{cases} \min\left\{f(y)/f(x), 1\right\} & \text{if } f(x) > 0, \\ 1 & \text{if } f(x) = 0. \end{cases}$$

- If f is heavy-tailed, then the extremal index is zero.
- Various versions of f and h can model various rates of increase of maxima.

What if Markov chain starts at a point? (J. and Truszczyński, SPL(2018))

- Let $\{Y_n\}$ be a positive Harris and aperiodic chain taking values in $\big(\mathbb{S}, \mathcal{S}\big)$ and with a stationary distribution π.
- Let $f : \big(\mathbb{S}, \mathcal{S}\big) \to \big(\mathbb{R}^1, \mathcal{B}^1\big)$ be measurable. Define $X_n = f(Y_n)$.
- If $\{X_n\}$ admits a continuous phantom distribution function G under some initial distribution λ, i.e. if we have

$$\sup_{x \in \mathbb{R}^1} \Big| \mathbb{P}_\lambda\big(M_n \leqslant x\big) - G^n(x) \Big| \to 0, \text{ as } n \to \infty, \qquad (*)$$

  then G is also a continuous phantom distribution function for the stationary (under π) sequence $\{X_n\}$.
- Conversely, if $\{X_n\}$ admits a continuous phantom distribution function G under π, then there exists a set $\mathbb{S}_0 \in \mathcal{S}$ satisfying $\pi\big(\mathbb{S}_0\big) = 0$ and such that relation $(*)$ holds for every initial distribution λ with the property that $\lambda(\mathbb{S}_0) = 0$

Hüsler's example and periodic non-stationarity

- Let $Y_1, Y_2, \ldots$ be i.i.d. and have the exponential distribution with parameter λ F_λ. Choose $\alpha > 0$ and set

$$X_1 = Y_1, X_2 = Y_2 + \alpha, X_3 = Y_3, X_4 = Y_4 + \alpha, \ldots.$$

- Hüsler in JApplProb(1986) observed that

$$\mathbb{P}(M_n \leqslant \log n + x) \to \exp(-\exp(-x + \alpha)/2).$$

- Hüsler used this fact to illustrate the formalism developed in his paper.
- We can simply say that $\{X_n\}$ admits a phantom distribution function

$$G(x) = \Big(F_\lambda(x) F_\lambda(x - \alpha)\Big)^{1/2}.$$

- Notice the geometric averaging in the above formula.

Hüsler's example and periodic non-stationarity

- In fact, when formulated in terms of phantom distribution functions, Hüsler's example is valid for any distribution function F such that $F^n(v_n) \to \gamma_1$, $F^n(v_n - \alpha) \to \gamma_2$, $0 < \gamma_1, \gamma_2 < 1$, for some sequence of levels v_n.
- Let us consider an important extension. Let $\{\alpha_k\}$ be a periodic sequence of numbers (i.e. $\alpha_{k+m \cdot p} = \alpha_k$ for all $k, m \in \mathbb{N}$ and some $p \in \mathbb{N}$).
- Let $\{Y_j\}$ be i.i.d. with distribution function F satisfying regularity conditions $F^n(v_n - \alpha_k) \to \gamma_k$, $0 < \gamma_1, \gamma_2, \ldots, \gamma_p < 1$.
- If we set $X_k = Y_k + \alpha_k$, then

$$G(x) = \Big(F(x - \alpha_1)F(x - \alpha_2)\ldots F(x - \alpha_p)\Big)^{1/p}$$

  is a phantom distribution function for $\{X_n\}$.
- Averaging! Also: undefinable extremal index.

Asymptotic Independent Representation for Maxima

- Let $\{X_k\}_{k\in\mathbb{N}}$ be a sequence of random variables. Define $M_n = \max_{0<k\leqslant n} X_k$.
- Suppose one can find a sequence $\{\widetilde{X}_k\}_{k\in\mathbb{N}}$ of independent random variables such that

$$\sup_{x\in\mathbb{R}^1} |\mathbb{P}(M_n \leqslant x) - \mathbb{P}(\widetilde{M}_n \leqslant x)| \to 0 \quad \text{as} \quad n \to \infty,$$

  where $\widetilde{M}_n$ is the n-th partial maximum of $\{\widetilde{X}_k\}$.
- We will say that $\{X_k\}_{k\in\mathbb{N}}$ admits an asymptotic independent representation for maxima (AIRM) $\{\widetilde{X}_k\}_{k\in\mathbb{N}}$.
- Clearly, if $\{\widetilde{X}_k\}$ are identically distributed, then their common distribution function G is a phantom distribution function for $\{X_k\}_{k\in\mathbb{N}}$.

The tool (Theorem 2, J., AoP(1993))

- Assume there is a non-decreasing sequence $\{v_n\}$ such that

$$\mathbb{P}(M_{[nt]} \leqslant v_n) \longrightarrow \exp(-\beta_t), \text{ as } n \to \infty, \quad t \geqslant 0,$$

  where the function β_t is continuous on $[0, \infty)$, $\beta_0 = 0$, $\lim_{t\to\infty} \beta_t = +\infty$.
- If the function β_t is of the form

$$\beta_t = h(\log t),$$

  where $h : (-\infty, +\infty) \to [0, +\infty)$ is convex, then $\{X_n\}$ admits an AIRM $\{\widetilde{X}_n\}$.
- $\{\widetilde{X}_n\}$ can be defined via its marginals

$$\widetilde{X}_k \sim F_k(x) = \begin{cases} 0, & \text{if } x < v_1; \\ \exp\left(\beta_{(k-1)/n} - \beta_{k/n}\right), & \text{if } v_n \leqslant x < v_{n+1}; \\ 1, & \text{if } x \geqslant \sup_n v_n. \end{cases}$$

Some comments on the main tool

- If $h(x) = B\exp(x)$, then $\beta_t = h(\log t) = B \cdot t$ and for each k

$$F_k(x) = \exp\left(\beta_{(k-1)/n} - \beta_{k/n}\right) = \exp(-B)^{1/n}, \text{ if } v_n \leqslant x < v_{n+1},$$

  i.e. F_k does not depend on k.
- In other words, if

$$\mathbb{P}(M_{[nt]} \leqslant v_n) \longrightarrow \exp(-B \cdot t), \text{ as } n \to \infty, \quad t \geqslant 0,$$

  then $\{X_n\}$ admits a phantom distribution function, independently of being stationary or non-stationary.
- If the function β_t is discontinuous, it may be uninformative. Let

$$F(x) = \begin{cases} 1 - x^{-\beta} & \text{for } x \geqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

  If $\{Y_k\}$ are i.i.d. with $\mathcal{L}(Y_k) \sim F$, define $X_k = k^{-1/\beta} Y_k$ and $v_n = \log^{1/\beta} n$. Then for *every* $t > 0$,

$$\mathbb{P}(M_{[nt]} \leqslant v_n) \longrightarrow e^{-1}.$$

Non-periodic drifts

- Let $Y_1, Y_2, \ldots$ be i.i.d. and have the Gumbel distribution: $F(x) = \exp(-\exp(-x))$. Let α_k be any numbers. Set $X_1 = Y_1 + \alpha_1, X_2 = Y_2 + \alpha_2, \ldots$.
- We have

$$\mathbb{P}(X_k \leqslant x) = \mathbb{P}(Y_k + \alpha_k \leqslant x) = F(x - \alpha_k) = \big(F(x)\big)^{e^{\alpha_k}}.$$

- Therefore

$$\mathbb{P}(M_{[nt]} \leqslant v_n) = \big(F(v_n)^n\big)^{(1/n)\sum_{k=1}^{[nt]} e^{\alpha_k}} \to exp(-B \cdot t),$$

  if $F(v_n)^n \to e^{-1}$ and

$$\frac{1}{n}\sum_{k=1}^{[nt]} e^{\alpha_k} \to B \cdot t, \quad t > 0.$$

- Distributions of $X_1, X_2, \ldots$ form so-called F^{α}-scheme, studied by Young, Weissmann, Nevzorov, Doukhan, ...

Other possible representations

- If $\{X_n\}$ are independent, and for some continuous β_t

$$\mathbb{P}(M_{[nt]} \leqslant v_n) \longrightarrow \exp(-\beta_t), \text{ as } n \to \infty, \quad t \geqslant 0,$$

  then necessarily $\beta_t = h(\log t)$ for some convex h.
- Are there any other interesting functions β_t?
- Surely: set $h(x) = B\exp(C \cdot x)$, $B, C > 0$. Then $\beta_t = h(\log t) = Bt^C$.
- Returning to the previous F^α-scheme we have

$$\mathbb{P}(M_{[nt]} \leqslant v_n) = \left(F(v_n)^{n^C}\right)^{(1/n^C)\sum_{k=1}^{[nt]} e^{\alpha_k}} \to exp(-B \cdot t^C),$$

  if $F(v_n)^{n^C} \to e^{-1}$ and

$$\frac{1}{n^C}\sum_{k=1}^{[nt]} e^{\alpha_k} \to B \cdot t^C, \quad t > 0.$$