Test for independence of long-range dependent time series using distance covariance

Herold Dehling (Ruhr-University Bochum)
(joint work with Annika Betken (University of Twente))

Paul's first letter to Herold, 16 September 1983

UNIVERSITE DE PARIS-SUD

EQUIPE DE RECHERCHE ASSOCIEE AU C.N.R.S. 532
STATISTIQUE APPLIQUEE
MATHEMATIQUE (BÂTI. 425)

91405 ORSAY CEDEX
TEL: 941.77.96

Paul Doukhan Frédéric Portal

Orsay, the 16/9/1983

Dear Professor Dehling,

Sincerely your's

Doukhan, Portal (1983)

PREPUBLICATIONS

PRINCIPES D'INVARIANCE FAIBLES, AVEC VITESSE,
DANS UN CADRE MELANGEANT

PAUL DOUKHAN FREDERIC PORTAL

N° 83 T 28

Université de Paris-Sud
Département de Mathématique
Bât. 425
91405 ORSAY France

Doukhan, Portal (1983)

Principes d'Invariance Faibles, avec vitesses, dans un cadre mélangeant

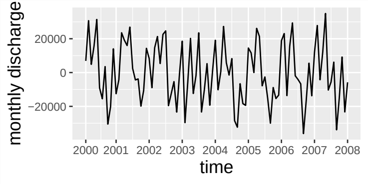
Paul Doukhan et Frédéric Portal

Table des matières

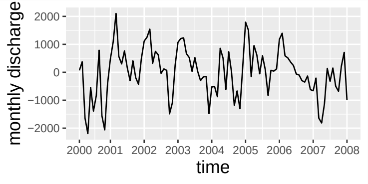
Test problem

Given: Observations $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$, viewed as the initial segment of a bivariate, stationary stochastic process $(X_k, Y_k)_{k \in \mathbb{N}}$.

Amazon River



Jutaí River



Goal: Decide on the testing problem

$$H_0\colon (X_k)_{k \in \mathbb{N}} \text{ and } (Y_k)_{k \in \mathbb{N}} \text{ are independent,}$$
$$H_1\colon (X_k)_{k \in \mathbb{N}} \text{ and } (Y_k)_{k \in \mathbb{N}} \text{ are dependent.}$$

Pearson's correlation coefficient

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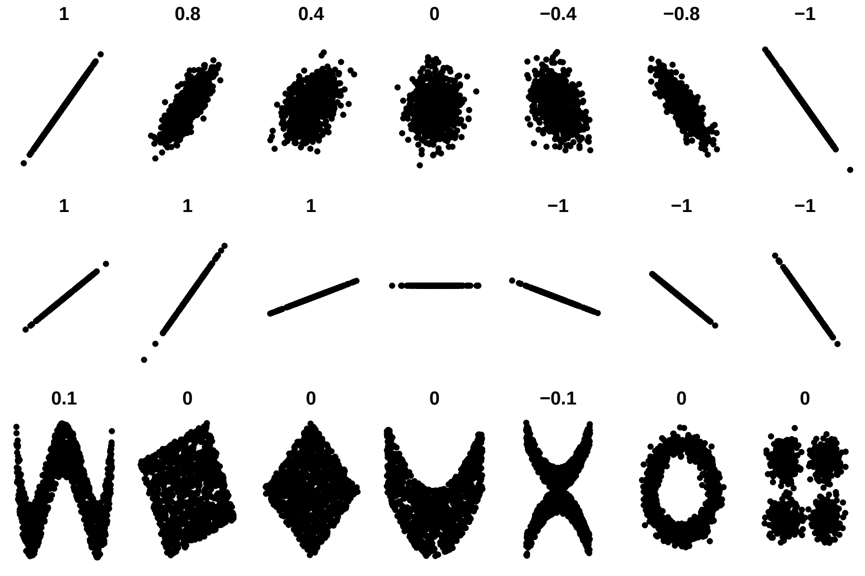
Pearson's correlation coefficient of a pair of random variables (X, Y):

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}}$$

Empirical correlation coefficient of n data pairs $(X_1, Y_1), \ldots, (X_n, Y_n)$:

$$r_{XY} = \frac{\sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^n (X_i - \bar{X})^2}\sqrt{\sum_{i=1}^n (Y_i - \bar{Y})^2}}$$

Pearson's correlation coefficient



Distance correlation

Characteristic function of a random variable X with distribution function F:

$$\varphi_X(t) = \mathrm{E}\exp(itX) = \int_{-\infty}^{\infty} \exp(itx)\,dF(x).$$

Joint characteristic function of two random variables X and Y:

$$\varphi_{X,Y}(s,t) = \mathrm{E}\exp(isX + itY).$$

Definition

Let $w : \mathbb{R}^2 \longrightarrow \mathbb{R}_+$ be a positive weight function. The distance covariance of two random variables X and Y is defined by

$$\begin{aligned} \mathrm{dcov}(X,Y) &:= \|\varphi_{X,Y} - \varphi_X\varphi_Y\|_w^2 \\ &:= \int_{\mathbb{R}}\int_{\mathbb{R}} \left|\varphi_{X,Y}(s,t) - \varphi_X(s)\varphi_Y(t)\right|^2 \; w(s,t)\,ds\,dt. \end{aligned}$$

For $w(s,t) = (\pi^2 s^2 t^2)^{-1}$: $\mathrm{dcov}(X,Y;w) = 0 :\Leftrightarrow X$ and Y are independent; *Székely, Rizzo, Bakirov (2007).*

Empirical distance covariance

Empirical characteristic function of a random variable X:

$$\varphi_X^{(n)}(t) = \frac{1}{n}\sum_{j=1}^{n} \exp(itX_j).$$

Joint empirical characteristic function of two random variables X and Y:

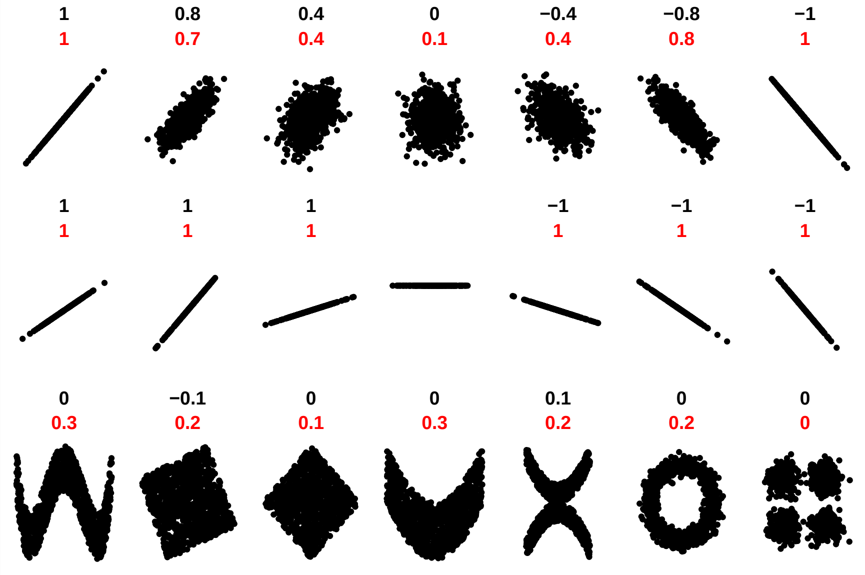
$$\varphi_{X,Y}^{(n)}(s,t) := \frac{1}{n}\sum_{j=1}^{n} \exp(isX_j + itY_j).$$

Definition

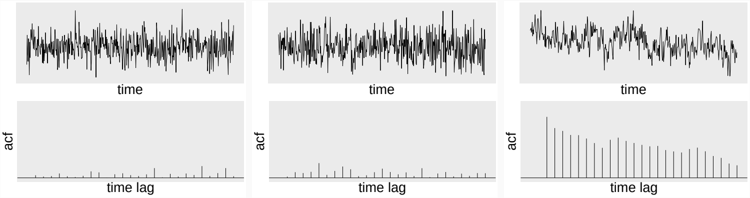
Empirical distance covariance/correlation of random variables X and Y:

$$\begin{aligned}
\mathrm{dcov}_n(X,Y) &:= \|\varphi_{X,Y}^{(n)}(s,t) - \varphi_X^{(n)}(s)\varphi_Y^{(n)}(t)\|_w^2 \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \left|\varphi_{X,Y}^{(n)}(s,t) - \varphi_X^{(n)}(s)\varphi_Y^{(n)}(t)\right|^2 w(s,t)\,ds\,dt, \\
\mathrm{dcorr}_n(X,Y) &:= \frac{\mathrm{dcov}_n(X,Y)}{\sqrt{\mathrm{dcov}_n(X,X)}\sqrt{\mathrm{dcov}_n(Y,Y)}}.
\end{aligned}$$

Pearson's correlation coefficient vs. distance correlation



Dependence in time series: IID, SRD, LRD



Characterization by the autocovariance function $\gamma(k) := \mathrm{Cov}(X_1, X_{k+1})$**:**

$$\sum_{k=1}^{\infty} |\gamma(k)| = 0 \qquad \sum_{k=1}^{\infty} |\gamma(k)| < \infty \qquad \sum_{k=1}^{\infty} |\gamma(k)| = \infty$$

V-statistic representation

Define the functions

$$f(s_1, s_2, s_3, s_4) := |s_1 - s_2| - |s_1 - s_3| - |s_2 - s_4| + |s_3 - s_4|$$
$$\tilde{h}((s_1, t_1), \ldots, (s_6, t_6))) := f(s_1, s_2, s_3, s_4)\, f(t_1, t_2, t_5, t_6).$$

Then we obtain for the empirical distance covariance with weight function $w(s, t) = 1/(s^2\, t^2)$ the V-statistic representation

$$\mathrm{dcov}_n(X, Y) = \frac{1}{n^6} \sum_{1 \leq i_1, \ldots, i_6 \leq n} \tilde{h}((X_{i_1}, Y_{i_1}), \ldots, (X_{i_6}, Y_{i_6})).$$

Note that the kernel is 1-degenerate, if the processes $(X_i)_{i\geq 1}$ and $(Y_i)_{i\geq 1}$ are independent, i.e. that

$$\mathrm{E}(f(x_1, X_2, X_3, X_4)\, f(y_1, Y_2, Y_5, Y_6)) = 0.$$

Large sample behavior via V-statistic theory: IID & SRD

IID processes $(X_i, Y_i)_{i\geq 1}$:

- Consistency by the LLN for V-statistics
- Limit distribution is a mixture of χ^2-distributions, if X_i and Y_i are independent: Székely, Rizzo, Bakirov (2007), Lyons (2013)
- Bootstrap procedure and consistency: D., Matsui, Mikosch, Samorodnitsky, Tafakori (2020)

SRD processes $(X_i, Y_i)_{i\geq 1}$:

- Consistency by the ergodic theorem for V-statistics: Davis, Matsui, Mikosch, Wan (2018)
- Limit distribution is a mixture of χ^2-distributions, if $(X_i)_{i\geq 1}$ and $(Y_i)_{i\geq 1}$ are independent: Davis et al. (2018), Kroll (2022).
- Bootstrap procedure and consistency: Betken, D., Kroll (2024)

For LRD processes, we use the representation of $\mathrm{dcov}_n(X, Y)$ as norm of an L_2-valued stochastic process.

Distance covariance for LRD processes

- Betken, D. (2024) study the asymptotic behavior of the empirical distance covariance when the underlying processes $(X_j)_{j\geq 1}$, $(Y_j)_{j\geq 1}$ are long-range dependent.
- The analysis is based on the original representation of the empirical distance covariance

$$\begin{aligned} \mathrm{dcov}_n(X,Y) &:= \|\varphi^{(n)}_{X,Y}(s,t) - \varphi^{(n)}_X(s)\varphi^{(n)}_Y(t)\|^2_w \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \left|\varphi^{(n)}_{X,Y}(s,t) - \varphi^{(n)}_X(s)\varphi^{(n)}_Y(t)\right|^2 w(s,t)\, ds\, dt, \end{aligned}$$

  and limit theorems for the process $\varphi^{(n)}_{X,Y}(s,t) - \varphi^{(n)}_X(s)\varphi^{(n)}_Y(t)$, viewed as elements of $L_2(\mathbb{R}^2, w(s,t)\, ds\, dt)$.
- Betken, D. (2024) propose a test based on empirical distance cross covariances that is consistent against arbitrary forms of dependence between the processes $(X_j)_{j\geq 1}$ and $(Y_j)_{j\geq 1}$.

Long-range dependence

Definition

A stationary time series $(X_n)_{n\in\mathbb{N}}$ is called long-range dependent (LRD), if

$$\gamma(k) := \mathrm{Cov}(X_1, X_k) = k^{-D}L(k),$$

where $0 < D < 1$ and L slowly varying; *Taqqu, Pipiras (2017)*.

- **fractional Brownian motion (fBm) with Hurst parameter** $0 < H < 1$: Gaussian process $B_H(t)$, $t \geq 0$, with $\mathrm{E}\, B_H(t) = 0$, $\mathrm{Var}\, B_H(t) = 1$, and $\mathrm{Cov}(B_H(t), B_H(s)) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$.
- **fractional Gaussian noise (fGn)**: $\xi_H(k) = B_H(k) - B_H(k-1)$, where $B_H(t)$, $t \geq 0$, is a fractional Brownian motion. Then

$$\gamma(k) = \frac{1}{2}\left(|k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H}\right) \sim H(2H-1)k^{-(2-2H)},\ k \to \infty,$$

i.e. for $H \in \left(\frac{1}{2}, 1\right)$ fractional Gaussian noise is long-range dependent with LRD parameter $D = 2 - 2H$.

Subordinated Gaussian processes

Time series $(X_k)_{k\in\mathbb{N}}$ with

$$X_k = G(\xi_k),$$

where

- $\mathrm{E}\, X_k = 0$,
- G measurable.
- $(\xi_k)_{k\in\mathbb{N}}$ is a stationary LRD Gaussian process with $\mathrm{E}\,\xi_1 = 0$, $\mathrm{Var}\,\xi_1 = 1$,

$$\gamma(k) = \mathrm{Cov}(\xi_1, \xi_k) = k^{-D} L(k),$$

  where $0 < D < 1$ (LRD parameter), L slowly varying.

Hypothesis test for independence

Goal: Decide on the testing problem

$$H_0\colon (X_k)_{k\in\mathbb{N}},\ (Y_k)_{k\in\mathbb{N}} \text{ are independent,}$$
$$H_1\colon (X_k)_{k\in\mathbb{N}},\ (Y_k)_{k\in\mathbb{N}} \text{ are dependent.}$$

Note: Tests based on $\mathrm{dcov}_n(X, Y)$ will only be consistent against alternatives where X_i and Y_i are dependent.

Tool: Choose as test statistic

$$\sum_{h=0}^{\infty} a_h \,\mathrm{dcov}_n(X, Y; h),$$

where $a_h \geq 0$,

$$\mathrm{dcov}_n(X, Y; h) := \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \varphi^{(n)}_{X,Y;h}(s,t) - \varphi^{(n)}_X(s)\varphi^{(n)}_Y(t) \right|^2 w(s,t)\,ds\,dt,$$

$$\varphi^{(n)}_{X,Y;h}(s,t) := \frac{1}{n} \sum_{j=1}^{n-h} e^{i(s\,X_j + t\,Y_{j+h})}.$$

Challenge

Given: Observations $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ stemming from real-valued time series $(X_k)_{k\in\mathbb{N}}$ and $(Y_k)_{k\in\mathbb{N}}$.

Goal: Decide on the testing problem

$$H_0\colon (X_k)_{k\in\mathbb{N}}, (Y_k)_{k\in\mathbb{N}} \text{ independent}, \quad H_1\colon (X_k)_{k\in\mathbb{N}}, (Y_k)_{k\in\mathbb{N}} \text{ dependent}.$$

Procedure: Reject H_0 for large values of the test statistic

$$T_n := \sum_{h=0}^{\infty} a_h \, \mathrm{dcov}_n(X, Y; h),$$

where $\mathrm{dcov}_n(X, Y; h) := \|\varphi^{(n)}_{X,Y;h}(s,t) - \varphi^{(n)}_X(s)\varphi^{(n)}_Y(t)\|^2_w$.

Intermediate goal: Fix critical values based on the statistic's distribution F_{T_n} under H_0.

Limit distribution under H_0

Theorem

$X_i = G_1(\xi_i)$, $Y_i = G_2(\eta_i)$, $i \geq 1$, where $(\xi_i)_{i\geq 1}$, $(\eta_i)_{i\geq 1}$ independent, LRD Gaussian processes with $\operatorname{E}(\xi_1) = \operatorname{E}(\eta_1) = 0$, $\operatorname{Var}(\xi_1) = \operatorname{Var}(\eta_1) = 1$, $\rho_\xi(k) = \operatorname{Cov}(\xi_1, \xi_{1+k}) = k^{-D_\xi} L_\xi(k)$ and $\rho_\eta(k) = \operatorname{Cov}(\eta_1, \eta_{1+k}) = k^{-D_\eta} L_\eta(k)$ for $D_\xi, D_\eta \in (0, 1)$ and L_ξ, L_η slowly varying. Assume that $\operatorname{E}|X_1| < \infty$, $\operatorname{E}|Y_1| < \infty$.

(i) If $D_\xi, D_\eta \in (\frac{1}{2}, 1)$

$$\sqrt{n}\left(\varphi^{(n)}_{X,Y;h}(s,t) - \varphi^{(n)}_X(s)\varphi^{(n)}_Y(t)\right) \xrightarrow{\mathcal{D}} Z_1(s,t),$$

(ii) If $G_1 = G_2 = \mathrm{id}$ and $D_\xi + D_\eta < 1$

$$n^{\frac{D_\xi + D_\eta}{2}} L_\xi^{-\frac{1}{2}}(n) L_\eta^{-\frac{1}{2}}(n)\left(\varphi^{(n)}_{X,Y;h}(s,t) - \varphi^{(n)}_X(s)\varphi^{(n)}_Y(t)\right) \xrightarrow{\mathcal{D}} Z_2(s,t),$$

where Z_1, Z_2 non-degenerate $L^2(\mathbb{R}^2, w(s,t)dsdt)$-valued stochastic processes and $\xrightarrow{\mathcal{D}}$ denotes convergence in $L^2(\mathbb{R}^2, w(s,t)dsdt)$, $w(s,t) = (\pi^2 s^2 t^2)^{-1}$; *Betken, D. (2021)*.

Limit distribution under H_0

Theorem

$X_i = G_1(\xi_i)$, $Y_i = G_2(\eta_i)$, $i \geq 1$, where $(\xi_i)_{i\geq 1}$, $(\eta_i)_{i\geq 1}$ independent, LRD Gaussian with $\mathrm{E}\,(\xi_1) = \mathrm{E}\,(\eta_1) = 0$, $\mathrm{Var}\,(\xi_1) = \mathrm{Var}\,(\eta_1) = 1$, $\rho_\xi(k) = \mathrm{Cov}(\xi_1, \xi_{1+k}) = k^{-D_\xi} L_\xi(k)$ and $\rho_\eta(k) = \mathrm{Cov}(\eta_1, \eta_{1+k}) = k^{-D_\eta} L_\eta(k)$ for $D_\xi, D_\eta \in (\frac{1}{2}, 1)$ and L_ξ, L_η slowly varying. Assume that $\mathrm{E}\,|X_1| < \infty$, $\mathrm{E}\,|Y_1| < \infty$.

Given a real-valued sequence a_h, $h \geq 0$, with $\sum_{h=0}^{\infty} |a_h| < \infty$

$$n \sum_{h=0}^{\infty} a_h \mathrm{dcov}_n(X, Y; h) \xrightarrow{\mathcal{D}} Z \sum_{h=0}^{\infty} a_h,$$

where

$$Z := \int_{\mathbb{R}^p} \int_{\mathbb{R}^q} \left|Z(s,t)\right|^2 w(s,t) ds\, dt,$$

and $(Z(s,t))_{s,t\in\mathbb{R}}$ a complex-valued Gaussian process; *Betken, D. (2024)*.

Outline of the proof I

Noting that $\phi_{X,Y}^{(n)}(s,t) = \frac{1}{n}\sum_{j=1}^{n} e^{i(sX_j + tY_j)}$, $\phi_X^{(n)}(s) = \frac{1}{n}\sum_{j=1}^{n} e^{isX_j}$, and $\phi_Y^{(n)}(s) = \frac{1}{n}\sum_{j=1}^{n} e^{itY_j}$, we obtain the decomposition

$$\begin{aligned}\phi_{X,Y}^{(n)}(s,t) - \phi_X^{(n)}(s)\phi_Y^{(n)}(t) = \frac{1}{n}\sum_{j=1}^{n}&(e^{isX_j} - \phi_X(s))(e^{isY_j} - \phi_Y(s)) \\ &- (\phi_X^{(n)}(s) - \phi_X(s))(\phi_Y^{(n)}(t) - \phi_Y(t))\end{aligned}$$

- For IID and for SRD data, the first of these terms is dominating, and

$$\sqrt{n}(\phi_{X,Y}^{(n)}(s,t) - \phi_X^{(n)}(s)\phi_Y^{(n)}(t)) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}(e^{isX_j} - \phi_X(s))(e^{isY_j} - \phi_Y(s)) + o_P(1)$$

- For LRD data, both terms might contribute, depending on the values of the LRD parameters D_ξ and D_η.

Outline of the proof II

Univariate Hermite expansion: For a Gaussian LRD process $(\xi_j)_{j\geq 1}$, and an L_2-function H, the asymptotic distribution of the partial sum $\sum_{j=1}^{n} H(\xi_j)$ is dominated by the first order non-zero term in the Hermite expansion of H,

$$H(x) = \sum_{q=m}^{\infty} \frac{J_q}{q!} H_q(x),$$

where, $H_q(x) = (-1)^q \frac{d^q}{dx^q} e^{-x^2/2}$ is the q-th order Hermite polynomial and

$$J_q = \mathrm{E}(H(\xi)\, J_q(\xi))$$

Bivariate Hermite expansion: The asymptotic distribution of $\sum_{j=1}^{n} H(\xi_j, \eta_j)$ is dominated by the lowest order non-zero term in the Hermite expansion

$$H(x, y) = \sum_{r=m}^{\infty} \sum_{p+q=r} \frac{J_{p,q}}{p!q!} H_p(x) H_q(y),$$

where $J_{p,q} = \mathrm{E}(H(\xi_1, \eta_1) H_p(\xi_1) H_q(\eta_1))$.

Outline of the proof III

Recall the decomposition

$$\phi_{X,Y}^{(n)}(s,t) - \phi_X^{(n)}(s)\phi_Y^{(n)}(t) = \frac{1}{n}\sum_{j=1}^{n}(e^{isX_j} - \phi_X(s))(e^{isY_j} - \phi_Y(s))$$
$$- (\phi_X^{(n)}(s) - \phi_X(s))(\phi_Y^{(n)}(t) - \phi_Y(t))$$

For the first term we need to consider the function

$$H(x,y) = (e^{isG_1(x)} - \phi_X(x))(e^{itG_2(y)} - \phi_Y(t)).$$

The lowest order term in the Hermite expansion is $H_1(s)\,H_1(y) = x\,y$; thus

$$\sum_{j=1}^{n}(e^{isX_j} - \phi_X(s))(e^{isY_j} - \phi_Y(s)) \approx \sum_{j=1}^{n}\xi_j \cdot \eta_j$$

The process $(\xi_j \cdot \eta_j)_{j\geq 1}$ has covariance function $r_{\xi\eta}(k) = k^{-(D_\xi + D_\eta)}L_\xi(k)L_\eta(k)$.

- $D_\xi + D_\eta > 1$: $\sqrt{n}\,(\phi_{X,Y}^{(n)}(s,t) - \phi_X^{(n)}(s)\phi_Y^{(n)}(t)) \to Z(s,t)$
- $D_\xi + D_\eta < 1$: $n^{(D_\xi + D_\eta)/2}L_\xi^{-1/2}L_\eta^{-1/2}(\phi_{X,Y}^{(n)}(s,t) - \phi_X^{(n)}(s)\phi_Y^{(n)}(t)) \to Z(s,t)$

Outline of the proof IV: $D_\xi + D_\eta > 1$

- If $D_\xi + D_\eta > 1$, the process $f_{s,t}(X_j, Y_j) = (e^{isX_j} - \phi_X(s))(e^{isY_j} - \phi_Y(s))$ is in the SRD regime, and

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} (e^{isX_j} - \phi_X(s))(e^{isY_j} - \phi_Y(s)) \longrightarrow Z(s,t),$$

  where $(Z(s,t))$ is a complex-valued Gaussian process; see Arcones (1994).

- Using techniques from Cremers and Kadelka (1986), one can show that convergence holds in $L_2(\mathbb{R}^2, w(s,t)dsdt)$.
- Under the above assumption

$$\sqrt{n}(\phi_X^{(n)}(s) - \phi_X(s))(\phi_Y^{(n)}(t) - \phi_Y(t)) \longrightarrow 0.$$

Outline of the proof V: $D_\xi + D_\eta < 1$

Note: Here, we only have results for Gaussian processes, i.e. $G(x) = x$.
In this case, both terms in the decomposition contribute to the limit:

$$(\varphi_X^{(n)}(s) - \varphi_X(s))(\varphi_Y^{(n)}(t) - \varphi(t)) \tag{1}$$

$$= -s\, t\, e^{-(s^2+t^2)/2}\Big(\frac{1}{n}\sum_{j=1}^{n} X_j\Big)\Big(\frac{1}{n}\sum_{j=1}^{n} Y_j\Big) + o_P\big(n^{-(D_X+D_Y)/2} L_X^{1/2}(n) L_Y^{1/2}(n)\big)$$

$$\frac{1}{n}\sum_{j=1}^{n}(e^{isX_j} - \phi_X(s))(e^{isY_j} - \phi_Y(s)) \tag{2}$$

$$= s\, t\, e^{-(s^2+t^2)/2}\frac{1}{n}\sum_{j=1}^{n} X_j Y_j + o_P\big(n^{-(D_X+D_Y)/2} L_X^{1/2}(n) L_Y^{1/2}(n)\big)$$

In the end, we obtain

$$n^{\frac{D_X+D_Y}{2}} L_X^{-1/2}(n) L_Y^{-1/2}(n)(\phi_{X,Y}^{(n)}(s,t) - \phi_X^{(n)}(s)\phi_Y^{(n)}(t))$$

$$= s\, t\, e^{-(s^2+t^2)/2}\Big(n^{-1+\frac{D_X+D_Y}{2}}\sum_{j=1}^{n} X_j\, Y_j + n^{-2+\frac{D_X+D_Y}{2}}\big(\sum_{j=1}^{n} X_j\big)\big(\sum_{j=1}^{n} Y_j\big)\Big) + o_P(1).$$

Subsampling: Sampling-window method

Given: Data $(X_1, Y_1), \ldots, (X_n, Y_n)$, statistic $T_n := T_n(X_1, \ldots, X_n, Y_1, \ldots, Y_n)$.

Problem: The limit distribution F_{T_n} of T_n is unknown.

Solution: Approximate the distribution of T_n by resampling.

1. Choose blocklength $l = \mathcal{O}\left(n^{\beta}\right)$, $0 < \beta < 1$, and lag $d = o(n)$.
2. Consider subsamples $B_{k,l} := (X_k, \ldots, X_{k+l-1}),\ C_{k,l} := (Y_k, \ldots, Y_{k+l-1})$, where $k = 1, \ldots, N$, $N = n - l + 1$.
3. Calculate $T_{l,k} = T_l\left(X_k, \ldots, X_{k+l-1}, Y_{k+d}, \ldots, Y_{k+d+l-1}\right)$, $k = 1, \ldots, m$, where $m = n - l - d$.
4. The empirical distribution function of $T_{l,1}, \ldots, T_{l,m}$

$$\widehat{F}_{m,l}(x) = \frac{1}{m}\sum_{i=1}^{m} 1_{\{T_{l,i} \leq x\}}$$

is taken as an estimator for the distribution function F_{T_n} of T_n.

Subsampling: Consistency in the LRD case

Consider

$$\widehat{F}_{m,l}(x) = \frac{1}{m}\sum_{i=1}^{m} 1_{\{T_{l,i} \leq x\}}, \; T_{l,k} = T_l\left(X_k, \ldots, X_{k+l-1}, Y_{k+d}, \ldots, Y_{k+d+l-1}\right),$$

where $m = n - l - d$, as an estimator for the distribution function F_{T_n} of T_n.

Theorem

$(X_n)_{n\in\mathbb{N}}$, $(Y_n)_{n\in\mathbb{N}}$ independent, stationary subordinated Gaussian LRD time series with LRD parameters D_X, D_Y. $T_n = T_n(X_1, \ldots, X_n, Y_1, \ldots, Y_n)$ statistics with cdf F_{T_n} such that $T_n \xrightarrow{\mathcal{D}} T$. Choose $d = o(n)$ and let $l = \mathcal{O}\left(n^{(1+min(D_X,D_Y))/2-\varepsilon}\right)$ for some $\varepsilon > 0$ and $m = n - l - d$. Then, under some technical assumptions, we have

$$\hat{F}_{m,l}(t) - F_{T_n}(t) \xrightarrow{\mathcal{P}} 0$$

for all points of continuity t of F_T. If F_T is continuous, then convergence is uniform; *Betken, D. (2024) & Betken, Wendler (2018).*

Consistency

Theorem

Assume that $(X_j, Y_j)_{j\geq 1}$ is a stationary ergodic process, and that $\mathsf{E}\,|X_1| < \infty$, $\mathsf{E}\,|Y_1| < \infty$. Then, as $n \to \infty$,

$$\mathrm{dcov}_n(X, Y; h) \longrightarrow \mathrm{dcov}(X, Y; h),$$

almost surely; *Kroll (2022) & Betken, D. (2024)*.

Theorem

Let $(X_j, Y_j)_{j\geq 1}$ be a stationary ergodic process such that, for some lag h, the random variables X_j and Y_{j+h} are dependent. Assume $\mathsf{E}\,|X_1|, \mathsf{E}\,|Y_1| < \infty$. Given a real-valued sequence $(a_h)_{h\geq 0}$, with $a_h > 0$,

$$\sum_{h=0}^{\infty} a_h \,\mathrm{dcov}_n(X, Y; h) \to \sum_{h=0}^{\infty} a_h \,\mathrm{dcov}(X, Y; h) > 0,$$

as $n \to \infty$, almost surely; *Betken, D. (2024).*

Simulations I

“Linearly” correlated data

$(X_1, \dots, X_n, Y_1 \dots, Y_n) := (2\Phi(Z_1) - 1, \dots, 2\Phi(Z_{2n}) - 1)$, where $(Z_1, \dots, Z_{2n})$ is multivariate normally distributed with mean 0 and covariance matrix

$$\Sigma = \begin{pmatrix} \tilde{\Sigma} & r\tilde{\Sigma} \\ r\tilde{\Sigma} & \tilde{\Sigma} \end{pmatrix},$$

where

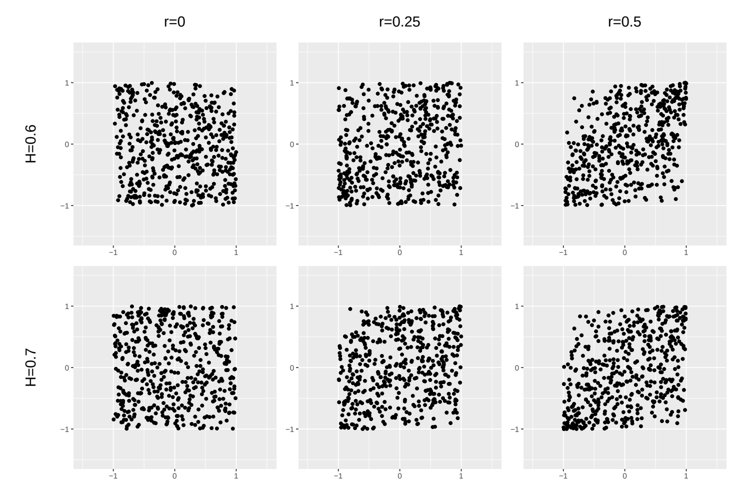
$$\tilde{\Sigma} = \begin{pmatrix} \rho(0) & \rho(1) & \dots & \rho(n-1) \\ \rho(1) & \rho(0) & \dots & \rho(n-2) \\ \vdots & \vdots & \ddots & \vdots \\ \rho(n-1) & \rho(n-2) & \dots & \rho(0) \end{pmatrix}$$

and

$$\rho(k) = \frac{1}{2}\left(|k+1|^{2H} - 2\,|k|^{2H} + |k-1|^{2H}\right).$$

Simulations I

“Linearly” correlated data



Simulations I

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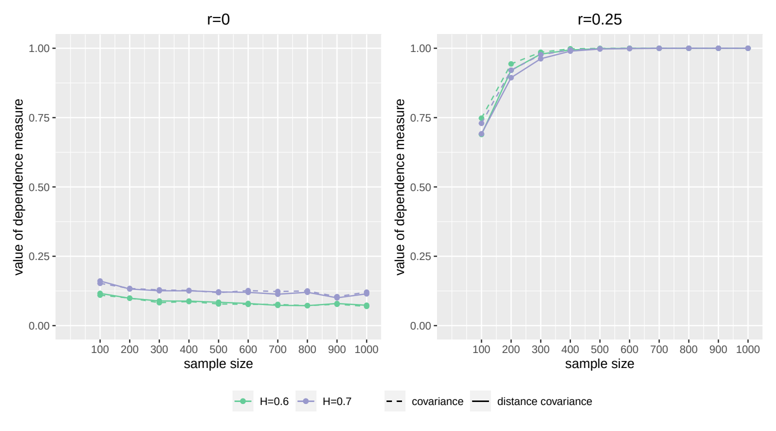


Simulations II

“Parabolically” correlated data

$(X_1, \dots, X_n, Y_1 \dots, Y_n)$, where

- $(X_1, \dots, X_n) := (2\Phi(Z_1) - 1, \dots, 2\Phi(Z_n) - 1)$, for fractional Gaussian noise $(Z_k)_{k \in \mathbb{N}}$ with parameter H,
- $(Y_1, \dots, Y_n)$ with

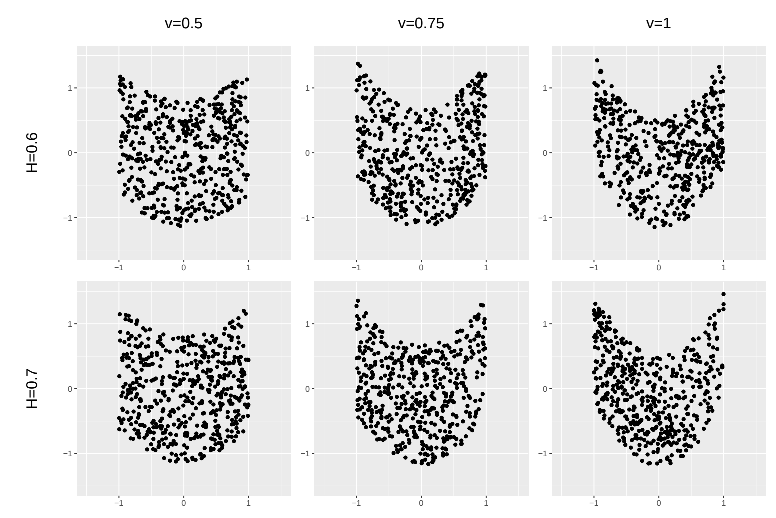
$$Y_i := v\left(X_i^2 - \frac{1}{3}\right) + w\xi_i, \;\; w = \sqrt{1 - \frac{4}{15}v^2},$$

  where $\xi_1, \dots, \xi_n$ are independent uniformly on $[-1, 1]$ distributed random variables.

(The choice of the parameter w guarantees $\operatorname{E} Y_1 = \operatorname{E} X_1 = 0$ and $\operatorname{Var}(Y_1) = \operatorname{Var}(X_1) = \frac{1}{3}$.)

Simulations II

"Parabolically" correlated data



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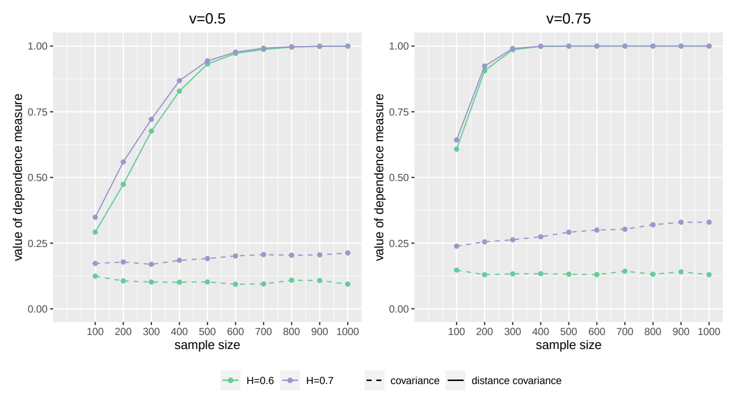

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Happy Birthday, Paul!