Mixing property and Spectral analysis of the Multivariate Hawkes process

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Abstract

The first result of this work is to extend the mixing property of the Hawkes process in [1] to the multivariate case by using the presentation of Hawkes as a Poisson cluster process. Other significant results are on the spectral analysis, they are given based on Bartlett spectrum [2]. Especially, the explicit expression of spectral densities function is available in the case random thinning process. This makes great sense when data is missing.

Multivariate Hawkes process

A **point process** is a random process whose realizations consist of event times falling along the line.

A **counting process** is a stochastic process $N(t) := \sum_{i\geq 1} \mathbb{I}_{\{T_i \leq t\}}$, associated with the point process (T_i). By convention $N_0 = 0$.

A **multivariate Hawkes process** can be defined as a list of d counting processes $\mathbf{N} = \{N_1, \cdots, N_d\}$ where the conditional intensity function of $N_j(\cdot)$ is defined by

$$\lambda_j^*(t) := \lim_{h\to 0} \frac{\mathbb{E}[N_j(t+h) - N_j(t)|\mathcal{H}_j(t)]}{h} = \eta_j + \sum_{i=1}^{d} \sum_{\{n: T_i^n < t\}} h_{ij}(t - T_i^n),$$

here for each N_j, $\mathcal{H}_j(\cdot)$ is the associated history, $\eta_j > 0$ is the *baseline intensity*, h_{ij} is the *reproduction function* and $\{T_i^n\}_n$ are the atoms of N_i. The multivariate Hawkes process **N** can also be seen as a cluster process.

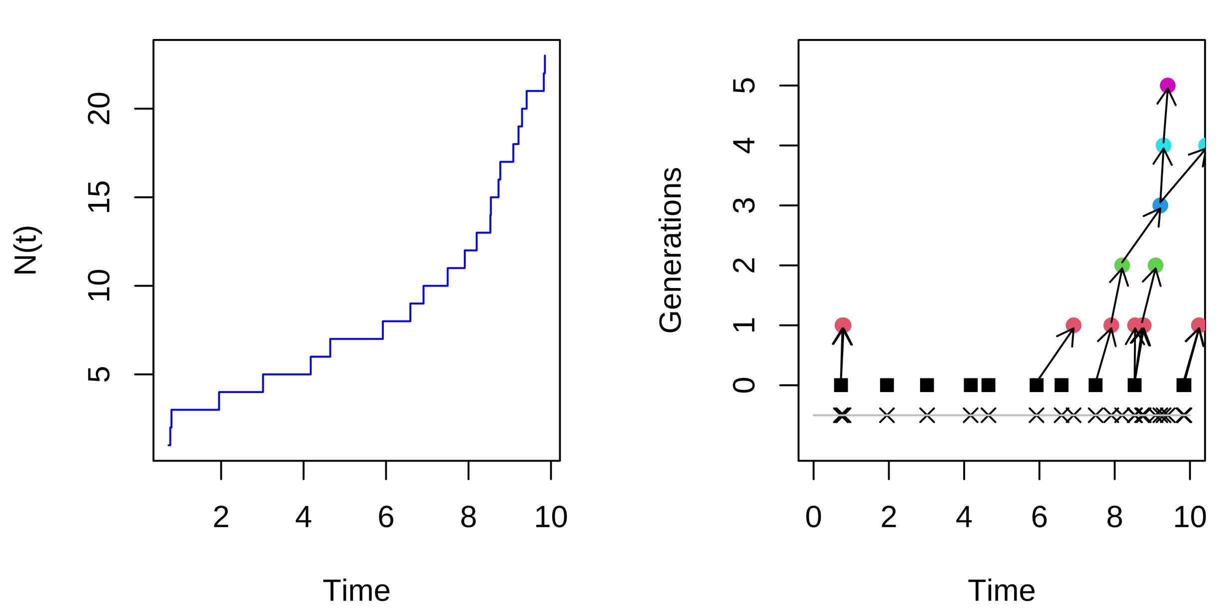

Fig. 1: An example of counting process (left) and Hawkes process represented as a collection of family trees (right).

Strong mixing and The Bartlett spectrum

The **strong mixing coefficient** can be defined as (see [5])

$$\alpha_{\mathbf{N}}(\tau) = \sup_{t\in\mathbb{R}} \sup_{\substack{\mathcal{A}\in\mathcal{E}_{-\infty}^{t} \\ \mathcal{B}\in\mathcal{E}_{t+\tau}^{\infty}}} \left|\mathrm{Cov}\left(\mathbb{I}_{\mathcal{A}}(\mathbf{N}), \mathbb{I}_{\mathcal{B}}(\mathbf{N})\right)\right|$$

where $\mathcal{E}_a^b$ is the σ-algebra generated by the cylinder sets on $(a, b]$, and $\mathbb{I}_{\mathcal{A}}(\mathbf{N})$ is the indicator function of the cylinder set $\mathcal{A}$, i.e $\forall B \in \mathcal{A}_{B,\mathbf{n}} = \{\mathbf{N} \in \mathfrak{N} : \mathbf{N}(B) = \mathbf{n}\}$, $\mathbb{I}_{\mathcal{A}}(\mathbf{N}) = 1$ if $\mathbf{N}(B) = \mathbf{n}$ and 0 otherwise. If $\alpha_{\mathbf{N}}(\tau) \to 0$ as $\tau \to \infty$, then the process is strong-mixing or α-mixing.

The Bartlett spectrum of **N** admits a matrix of spectral densities given by [2]

$$\left(\gamma_{ij}^{\mathbf{N}}(\omega)\right) = \left[\mathbf{I} - \left(\tilde{\mathbf{H}}(-\omega)\right)^{\intercal}\right]^{-1} \mathrm{diag}\,(m_1, \cdots, m_d)\, \left[\mathbf{I} - \tilde{\mathbf{H}}(\omega)\right]^{-1}$$

where $\boldsymbol{m} = (m_1, \cdots, m_d) = \mathbb{E}(\boldsymbol{\lambda}_t) = (\mathbf{I} - \mathrm{M})^{-1}\boldsymbol{\eta}$ denotes the vector of mean intensities of the process, $\mathrm{M} := (\|h_{ij}\|_1)$, and $\tilde{\mathrm{H}}$ the matrix of component-wise Fourier transforms of $\mathbf{H} = (h_{ij})$.

Result on strong mixing property

Let **N** be a multivariate Hawkes process with the spectral radius of the matrix $\mathbf{M} := (\|h_{ij}\|_1)$ is strictly less than 1 (for stationarity reason). Assume that there exists $\beta > 0$ such that

$$\nu_{1+\beta} = \sup_{1\leq i,j\leq d} \int_{\mathbb{R}} t^{1+\beta} h_{ij}^*(t)\mathrm{d}t < \infty \qquad (1)$$

where $h_{ij}^* = h_{ij}/\|h_{ij}\|_1$. Then, the process **N** is strong mixing. More precisely, polynomially mixing, i.e for any $0 < \gamma < \beta$,

$$\alpha_{\mathbf{N}}(\tau) = \mathcal{O}\left(\tau^{-\gamma}\right).$$

Main steps of proof

- Using the positive association of Hawkes processes [3] and extending the results of [5] to the multivariate case to evaluate the covariance of the indicator functions by the covariance of the counting process.
- We rewrite covariance of counting process to that of branching process.
- Using assumption on the reproduction kernel (1), Hölder's and Markov's inequalities, we derive an upper bound for the covariance.

Results on spectral densities functions

Bin-count process

A bin-count process with binsize $\Delta > 0$ associated to **N**, $\{\mathbb{X}_t\}_{t\in\mathbb{R}} = \left\{\mathbf{N}\big((t\Delta, (t+1)\Delta]\big)\right\}_{t\in\mathbb{R}}$ has spectral density functions given by

$$f_{\mathbb{X}}(\omega) = \Delta \mathrm{sinc}^2\left(\frac{\omega}{2}\right) \gamma_{ij}^{\mathbf{N}}\left(\frac{\omega}{\Delta}\right) \qquad (2)$$

Random sampling

Let $\mathbb{X}$ denote the process obtained from thinning **N** by a process $\mathcal{O}_t$, i.e $\mathbb{X}_B(t) = \int_B \mathcal{O}_t d(\mathbf{N}_t)$, where $\mathcal{O}_t = 1$ with probability m_1 and 0 with probability $1 - m_1$. Then,

$$f_{\mathbb{X}}(\omega) = \mathcal{F}\{R(u)\breve{C}_2^{\mathbf{N}}(\mathrm{d}u)\}(\omega) + \lambda_1^2\mathcal{F}\{R(u)\ell(\mathrm{d}u)\}(\omega) + m_1^2\mathcal{F}\{\breve{C}_2^{\mathbf{N}}(\mathrm{d}u)\}(\omega) \qquad (3)$$

where $\breve{C}_2^{\mathbf{N}}(\mathrm{d}u)$ is reduced covariance measure (see [2]), $\mathcal{F}$ is Fourier transform, $R(t)$ auto-covariance function of $(\mathcal{O}_t)$ and $\lambda_1 = \mathbb{E}(\mathbf{N}(0,1])$ is a determined constant.

Proofs

- (2) is directly obtained from [1, Section 4.1].
- The reduced moment can be related to that of the Hawkes process $\breve{M}_2(\mathrm{d}u) = \mathbb{E}(\mathcal{O}_0\mathcal{O}_u)\breve{M}_2^{\mathbf{N}}(\mathrm{d}u) = (R(u) + m_1^2)\breve{M}_2^{\mathbf{N}}(\mathrm{d}u)$. We then use the relation of reduced covariance and reduced second-moment and note that the density functions is the Fourier transform of reduced covariance measure [2].

Remarks and Perspectives

- The strong mixing property is pretty good (polynomial) to ensure an asymptotically normal estimate. Furthermore, the condition (1) is easy to fulfill.
- When the arrival times are not observed, we proposed a spectral approach for the estimation of the Hawkes process from their discrete-time counting series.
- The explicit and easily computable formulas are available for any stationary renewal process. They can be used for the estimation method in the case of missing or unobserved data.
- *Bootstrap* approach to inference in multivariate Hawkes process models.
- Question for Non causal/ non-linear/ non-stationary Hawkes processes.

Examples for exponential Hawkes process

We consider the exponential model, where reproduction function h is then defined as $h(t) = \mu\beta e^{-\beta t}$ (here $0 < \mu < 1$). Following Example 8.2(e) in [2], the Bartlett spectrum is

$$\gamma^N(\omega) = \lambda_1 \frac{\beta^2 + \omega^2}{\beta^2(1-\mu)^2 + \omega^2}$$

where $\lambda_1 = \eta(1-\mu)^{-1}$.

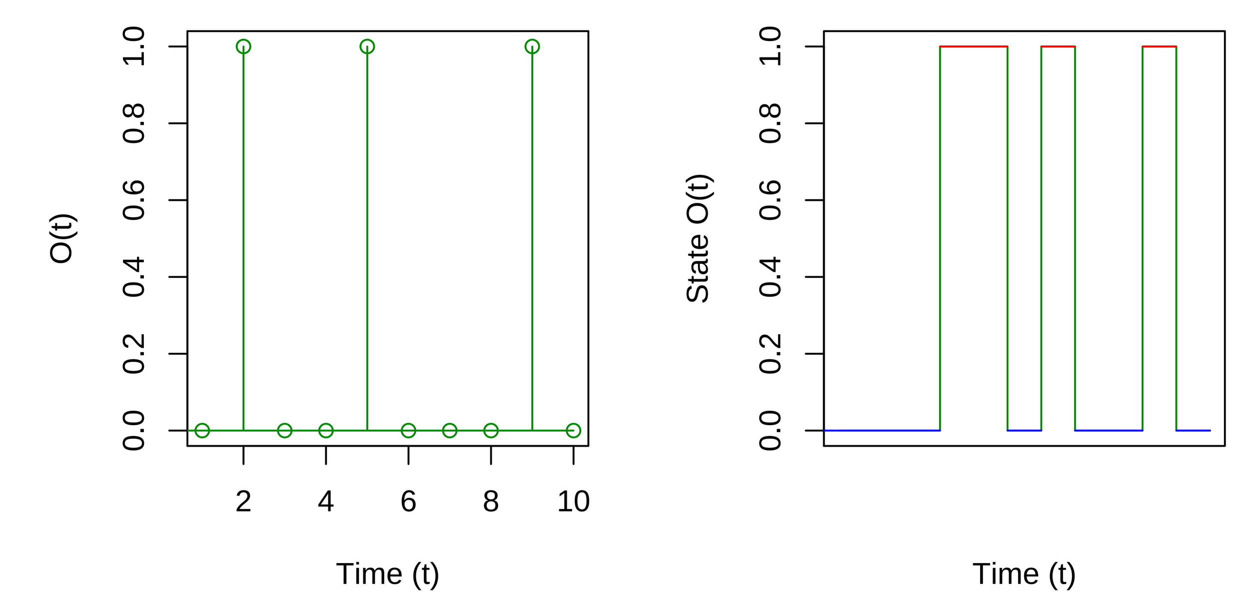

Fig. 2: An example $\mathcal{O}_t \sim$ Ber(p) (left) and alternating exponential distribution (right).

- **Example 1.** The process $\mathcal{O}_t$ has Bernoulli distribution parameters p then $R(t) = p(1-p)1_{\{t=0\}}(t)$. Therefore,

$$f_{\mathbb{X}}(\omega) = p\lambda_1 \left(1 - p + p\frac{\beta^2 + \omega^2}{\beta^2(1-\mu)^2 + \omega^2}\right). \qquad (4)$$

- **Example 2.** The duration of time t stays in state $k \in \{0, 1\}$ of $\mathcal{O}_t$ has exponential distribution with parameters β_0 and β_1, the spectral density is

$$f_{\mathbb{X}}(\omega) = S * \gamma^N(\omega) + \lambda_1^2 S(\omega) + m_1^2\gamma^N(\omega) \qquad (5)$$

where each term can be calculated precisely. Inspired by results in [4], we have

$$S(\omega) = \frac{2\beta_1\beta_0}{(\beta_1+\beta_0)\left(\omega^2 + (\beta_1+\beta_0)^2\right)},$$

$$S * \gamma(\omega) = \frac{\lambda_1\beta_0\beta_1}{\beta_0+\beta_1}\left(\frac{a}{\beta(1-\mu)} + \frac{b\omega + c}{\beta_1 + \beta_0}\right)$$

where a, b, c depend on $\beta, \mu, \beta_1, \beta_0$ and ω, and can be numerically computed.

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