On Estimating Operators of Functional Time Series

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Abstract

In functional time series analysis one is interested in estimating parameters which are often operators. This poster broadly outlines the estimation procedure for operators of well-known processes in separable Hilbert spaces under mild conditions using dimension reduction and Tikhonov regularization.

1 Introduction

When dealing with functional time series, it is often necessary to estimate their parameters. Usually, these parameters are elements of the given space or (linear) operators. Our aim is to outline the estimation procedure for operators of well-known functional time series, namely of AR(MA) and (G)ARCH processes in separable Hilbert spaces. We estimate these operators using certain Yule-Walker equations where specific lagged (cross-)covariance operators of the given processes come into play. Moreover, by utilizing dimension reduction, Tikhonov regularization, a Sobolev and a mild weak dependence condition, we derive the asymptotic behaviour of the estimation errors. The central ideas and results in this poster originate from the articles [5, 6].

2 Examples of processes in separable Hilbert spaces

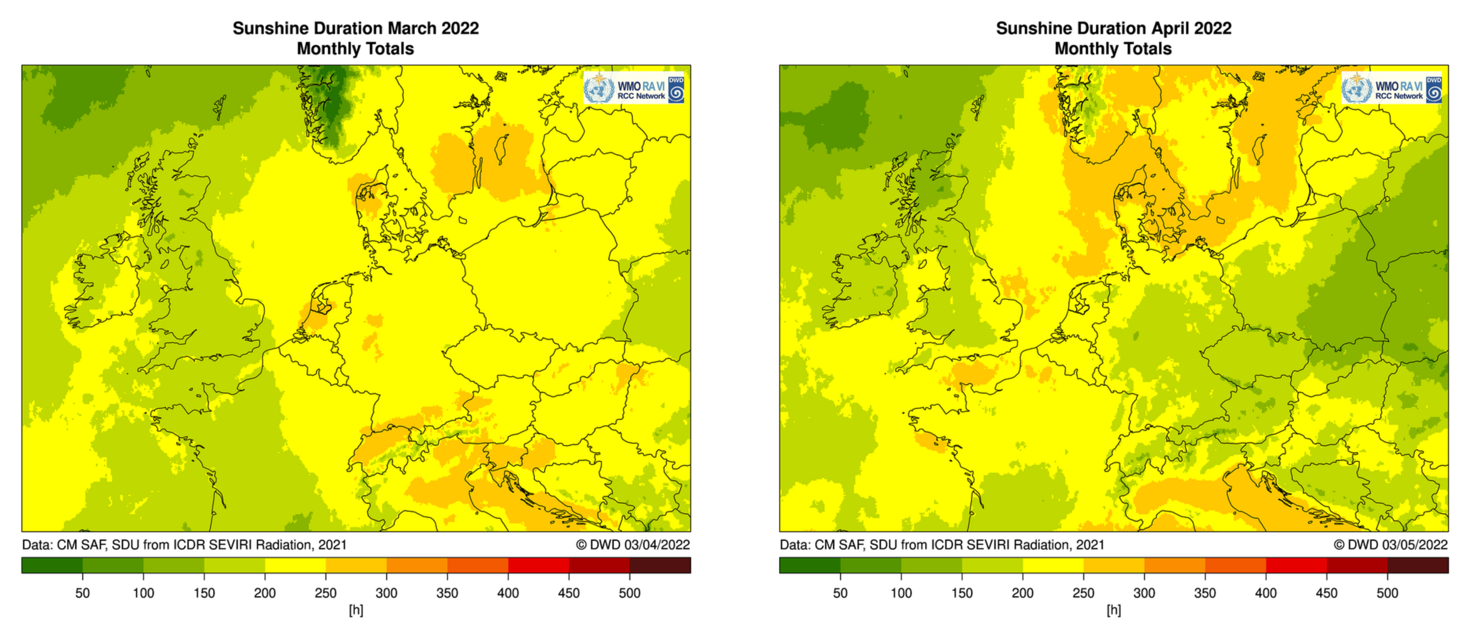

Figure 1: Graphs of the monthly sunshine duration in central europe in March and April 2022 (measured in hours/month per region), retrieved from www.dwd.de (German Meterological Service). These realizations can be interpreted as consecutive realizations of an $L^2[0,1]^2$-valued process.

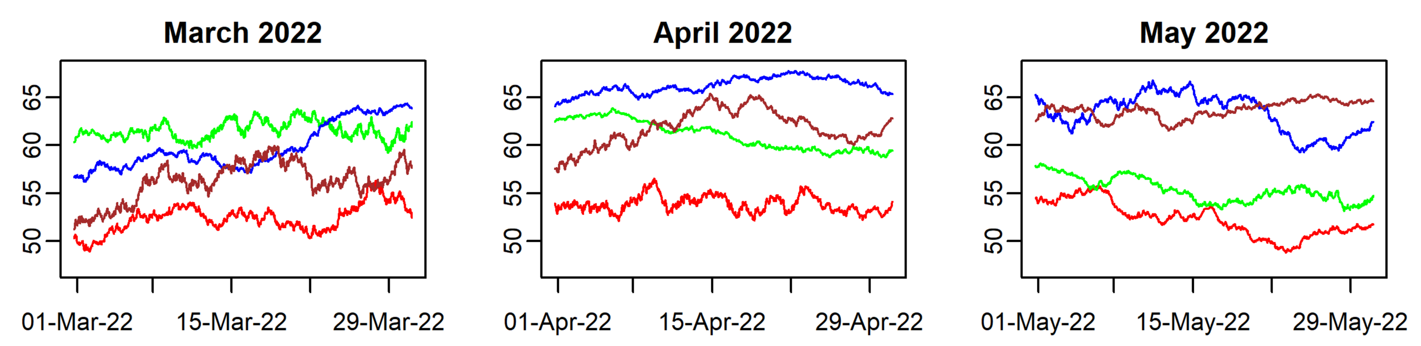

Figure 2: Three consecutive realizations of a fictitious process describing the share values of four assets of an portfolio, e.g., measured in *EUR*, being interpretable as consecutive realizations of an $(L^2[0,1])^4$-valued process.

3 Functional AR(MA) and (G)ARCH processes

3.1 Definitions

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ be a separable Hilbert space, and $(\varepsilon_k)_k$ an i.i.d., centered, $\mathcal{H}$-valued process. $(X_k)_k$ is an $\mathcal{H}$-valued, centered *autoregressive moving-average* (ARMA) process with orders $p, q \in \mathbb{N}$ if

$$X_k = \sum_{i=1}^{p} \alpha_i(X_{k-i}) + \sum_{j=1}^{q} \beta_j(\varepsilon_{k-j}) + \varepsilon_k, \quad \forall k, \tag{1}$$

where $\alpha_i, \beta_j \colon \mathcal{H} \to \mathcal{H}$ are bounded linear operators for all i, j. If $q = 0$ in (1), then $(X_k)_k$ is an *autoregressive* (AR) process with order p. Other important processes are functional *(generalized) autoregressive conditional heteroskedastic* ((G)ARCH) processes which allow the processes' variances to condition on the past (that is conditional heteroskedasticity). These processes are already defined in $L^2[0,1]$, see, e.g., [5]. There, $(X_k)_k$ is an $\mathcal{H}$-valued GARCH process with orders p, q if

$$X_k = \varepsilon_k \sigma_k, \quad \sigma_k^2 = \delta + \sum_{i=1}^{p} \alpha_i(X_{k-i}^2) + \sum_{j=1}^{q} \beta_j(\sigma_{k-j}^2), \quad \forall k, \tag{2}$$

with $\delta \in \mathcal{H}$ denoting a constant variance proportion, and where the product of two elements of $L^2[0,1]$ has to be interpreted pointwise.

3.2 Features

At least in $L^2[0,1]$, pointwise squared (G)ARCH are AR(MA) processes, see [5], which allows to derive estimators for the (G)ARCH parameters using AR(MA) equations. Hence, we consider AR(MA) processes, where we only investigate AR processes for the sake of simplicity.

Let $\boldsymbol{X} = (X_k)_k$ be an $\mathcal{H}$-valued, centered AR(p) process having finite second moments, with the operators $\alpha_1, \ldots, \alpha_p$ being elements of the space $\mathcal{S}_{\mathcal{H}} = \mathcal{S}_{\mathcal{H},\mathcal{H}}$ of Hilbert-Schmidt operators mapping from $\mathcal{H}$ to $\mathcal{H}$. Thus, $\boldsymbol{A}_p \coloneqq (\alpha_1, \ldots, \alpha_p)^T \in \mathcal{S}_{\mathcal{H}^p,\mathcal{H}}$, and the following *Yule-Walker equation*

$$\mathfrak{S}_{p,1} = \boldsymbol{A}_p \mathfrak{S}_p \tag{3}$$

is satisfied, where $\mathfrak{S}_{p,1} = \mathscr{C}_{X_p(p),X_{p+1}}$ is the cross-covariance operator of $X_p(p)$ and X_{p+1}, where $\boldsymbol{X}(p) = (X_k(p))_k$ is an $\mathcal{H}^p$-valued process derived from $(X_k)_k$, and where $\mathfrak{S}_p = \mathscr{C}_{0;\boldsymbol{X}(p)}$ denotes the autocovariance operator of $\boldsymbol{X}(p)$.

4 Estimation

Here, we estimate the operators of the $\mathcal{H}$-valued AR(p) process stated above.

4.1 Assumptions

- The innovation process $(\varepsilon_k)_k$ has finite, fourth moments
- $(X_k)_k$ is stationary and L^4-m-approximable, i.e., $(X_k)_k$ can be approximated by an m-dependent process, where the approximation errors measured with the norm of the space of the random variables having finite, fourth moments, are summable ($\to$ this weak dependence condition allows deriving consistency assertions for the (cross-)covariance and our AR operators)
- $\mathfrak{S}_p$ is injective ($\to$ $\boldsymbol{A}_p$ is identifiable from the Yule-Walker equation (3))
- For all eigenvalues of $\mathfrak{S}_p$ holds $c_j \neq c_{j+1}$ and $\kappa(j) = c_j$, with $\kappa\colon \mathbb{R} \to \mathbb{R}$ being a convex function ($\to$ the reciprocal spectral gaps are then well-defined)
- With $(\Phi_{i;j;p})_{i,j}$ being a complete orthonormal system (CONS) of $\mathcal{S}_{\mathcal{H}^p,\mathcal{H}}$, holds for some $\beta > 0$,

$$\sum_{i,j=1}^{\infty} \langle \boldsymbol{A}_p, \Phi_{ij;p}\rangle_{\mathcal{S}_{\mathcal{H}^p,\mathcal{H}}}^2 (1 + i^{2\beta} + j^{2\beta}) < \infty \tag{4}$$

($\to$ the approximation errors for the given vector of operators $\boldsymbol{A}_p$ decay sufficiently fast)

4.2 Estimation procedure and asymptotic results

We estimate the vector of AR operators $\boldsymbol{A}_p$ by using the Yule-Walker equation (3). The cross-covariance operator $\mathfrak{S}_{p,1}$ and the covariance operator $\mathfrak{S}_p$ are estimated by their moment estimators $\hat{\mathfrak{S}}_{p,1}$ resp. $\hat{\mathfrak{S}}_p$ based on a sample of $\boldsymbol{X} = (X_k)_k$. However, the direct estimation of $\boldsymbol{A}_p$ is problematic, as the inverse of the compact operator $\mathfrak{S}_p$ is not bounded. For this reason we make use of the *Tikhonov regularization* $\hat{\mathfrak{S}}_p^\dagger \coloneqq \hat{\mathfrak{S}}_p(\hat{\mathfrak{S}}_p^2 + \vartheta_N \mathbb{I}_{\mathcal{H}^p})^{-1}$, where $(\vartheta_N)_N \subseteq (0,\infty)$ is a sequence with $\vartheta_N \to 0$, and $\mathbb{I}_{\mathcal{H}^p}\colon \mathcal{H}^p \to \mathcal{H}^p$ the identity map. To estimate $\boldsymbol{A}_p$, we project on a K-dimensional space and use

$$\hat{\boldsymbol{A}}_p \coloneqq \hat{\mathfrak{S}}_{p,1} \hat{\mathfrak{S}}_p^\dagger \coprod_{\hat{\mathfrak{c}}_{p,1}}^{\hat{\mathfrak{c}}_{p,K}}, \tag{5}$$

where $\hat{\mathfrak{c}}_{p,1}, \ldots, \hat{\mathfrak{c}}_{p,K}$ denote the eigenfunctions of $\hat{\mathfrak{S}}_p$ associated to the first biggest eigenvalues $\hat{c}_{p,1} \geq \cdots \geq \hat{c}_{p,K}$, and $\coprod_{\hat{\mathfrak{c}}_{p,1}}^{\hat{\mathfrak{c}}_{p,K}}$ the operator projecting on $\mathrm{lin}\{\hat{\mathfrak{c}}_{p,1}, \ldots, \hat{\mathfrak{c}}_{p,K}\} \subseteq \mathcal{H}^p$.

Theorem 4.1. *Let the assumptions above hold, and let $\vartheta_N = \mathrm{O}(N^{-1/2})$. Moreover, $(\mathfrak{c}_j)_j \subseteq \mathscr{H}$ denotes the eigenfunction sequence of the autocovariance operator $\mathscr{C}_{0;\boldsymbol{X}}$.*

(a) *Let $\mathscr{J}_{p,K} \coloneqq \{\Phi_{i,j;p} | 1 \leq i, j \leq K\}, K \in \mathbb{N}$. If $\langle \boldsymbol{A}_p(\mathfrak{c}_{p,l}), \mathfrak{c}_j\rangle_{\mathscr{H}} = 0$ for all j, l with $l \leq K < j$,*

$$||\hat{\boldsymbol{A}}_p - \coprod_{\mathscr{J}_{p,K}} \boldsymbol{A}_p||_{\mathcal{S}_{\mathcal{H}^p,\mathcal{H}}}^2 = \mathrm{O}_{\mathbb{P}}(N^{-1}). \tag{6}$$

(b) *Let $K = K_N$ be an increasing sequence with $c_{p,K}^{-2}\gamma_{p,K}^2 K^{2\beta+1} = \mathrm{O}(N)$, where $\gamma_{p,K} \coloneqq (c_{p,K} - c_{p,K+1})^{-1}$ and $\sum_{l=1}^{K} (\frac{c_{p,l}^2}{c_{p,l}^2 + \vartheta_N})^2 \sum_{j>K} \langle \boldsymbol{A}_p(\mathfrak{c}_{p,l}), \mathfrak{c}_j\rangle_{\mathscr{H}}^2 = \mathrm{O}(K^{-2\beta})$. Then,*

$$||\hat{\boldsymbol{A}}_p - \boldsymbol{A}_p||_{\mathcal{S}_{\mathcal{H}^p,\mathcal{H}}}^2 = \mathrm{O}_{\mathbb{P}}(K^{-2\beta}). \tag{7}$$

5 Conclusions

This poster discusses the estimation of operators that appear in functional time series, to be precise, of operators in AR(MA) and (G)ARCH processes having values in separable Hilbert spaces. The estimation procedure uses dimension reduction, Tikhonov regularization, a Sobolev condition for the operators to be estimated and requires the given processes to satisfy a mild weak dependence condition. We derive the asymptotic behaviour of estimation errors for specific processes.

It would be nice to deduce these results also on general, separable Banach spaces. To derive estimators for operators of other parametric functional time series would also be of interest, as well as the asymptotic distribution of the estimation errors and Bernstein inequalities.

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