Connecting Spatial Statistics methods with the analysis of Stochastic Partial Differential Equations

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Motivations

An usual (basic) model in Spatial Statistics consists in a random field $(Z(x))_{x \in D \subset \mathbb{R}^d}$ with a particular covariance structure $C_Z(x,y) = \mathbb{C}ov(Z(x), Z(y))$ which can be fitted to spatialised data with different techniques. Recently, the so called *SPDE Approach* [5, 4] is of increasing interest. It consists in interpreting the random field as a solution to a Stochastic Partial Differential Equation. This allows us to:

- Obtain covariance structures with interesting properties from solutions of SPDEs.
- Give a physical interpretation for some covariance models.
- Analyze covariance models with PDE numerical tools: efficient simulations and inference techniques based on PDE-solvers (FEM, Fourier Analysis, etc).

Theoretical Framework

Generalized Random Fields: $Z = (\langle Z, \varphi \rangle)_{\varphi \in \mathscr{S}(\mathbb{R}^d)}$ where $\mathscr{S}(\mathbb{R}^d)$ is the Schwartz space. Z is described by its generalized covariance structure $C_Z \in \mathscr{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ (where $\mathscr{S}'$ denotes the dual of $\mathscr{S}$, the space of tempered distributions):

$$\mathbb{C}ov(\langle Z, \varphi \rangle, \langle Z, \phi \rangle) = \langle C_Z, \varphi \otimes \overline{\phi} \rangle.$$

Linear SPDEs: continuous linear operator $\mathcal{L} : \mathscr{S}' \to \mathscr{S}'$ (differentiation, Fourier Transform) can be applied to any random field Z. Given a random field X, we look at for a random field U such that:

$$\langle \mathcal{L}U, \varphi \rangle = \langle X, \varphi \rangle, \quad \forall \varphi \in \mathscr{S}(\mathbb{R}^d).$$

General Result: U satisfies this if and only if

$$\mathcal{L} \otimes \overline{\mathcal{L}} C_U = C_X = \mathcal{L} \otimes \mathcal{I} C_{U,X},$$

with $C_{U,X}$ the cross-covariance between U and X.

Existence and Uniqueness of Stationary Solutions

Stationary case: Z is *second-order stationary* if there exists $\rho_Z \in \mathscr{S}'(\mathbb{R}^d)$ such that $\langle C_Z, \varphi \otimes \overline{\phi} \rangle = \langle \rho_Z, \varphi * \check{\overline{\phi}} \rangle$. ρ_Z is the Fourier Transform of a positive slow-growing measure, $\rho_Z = \mathscr{F}(\mu_Z)$ (Bochner-Schwartz Theorem). μ_Z is the *spectral measure* of Z. **Example:** W random field with $\rho_W = \delta_0$, so $d\mu_W(x) = (2\pi)^{-d/2} dx$, is called a *White Noise*.

Consider equations of the type over $\mathbb{R}^d$

$$\mathcal{L}_g U = X$$

whit X stationary with spectral measure μ_X and $\mathcal{L}_g$ is a symbol-defined operator $\mathcal{L}_g(\cdot) := \mathscr{F}^{-1}(g \mathscr{F}(\cdot))$. g hermitian ($\check{\overline{g}} = g$), measurable and polynomially bounded.

Theorem 1. *[3] There exists stationary solutions to $\mathcal{L}_g U = X$ if and only if there exists $n \in \mathbb{N}$ such that $\int_{\mathbb{R}^d} \frac{d\mu_X(\xi)}{|g(\xi)|^2 (1 + |\xi|^2)^n} < \infty$. In such case, the solution is unique up-to-a-modification if and only if $|g| > 0$, with spectral measure $\mu_U = |g|^{-2} \mu_X$.*

Illustrations:

- Matérn Covariance (K-Bessel) [6, 5]: $\alpha \in \mathbb{R}$, $\kappa, \sigma > 0$. Unique solution to

$$(\kappa^2 - \Delta)^{\alpha/2} U = \sigma W.$$

- Advection-diffusion: $\kappa > 0$, $v \in \mathbb{R}^d, A \in \mathbb{R}^{d \times d}$ non-negative-definite. Unique solution to

$$\frac{\partial U}{\partial t} + \kappa^2 U + v^T \nabla U - \nabla \cdot A \nabla U = X.$$

- Heat Equation: $a > 0$, never unique solution to

$$\frac{\partial U}{\partial t} - a \Delta U = X$$

When $X = W$, existence only if $d \geq 3$.

- Wave Equation: $c > 0$. Never unique solution to

$$\frac{\partial^2 U}{\partial t^2} - \alpha \Delta U = X.$$

No solutions if $X = W$. If $X = 0$, the spatial covariance can be chosen freely.

Special illustration: First Order Evolution Models

We consider the Cauchy problem

$$\begin{cases} \frac{\partial U}{\partial t} + \mathcal{L}_g U = X \\ U_{t=0} = U_0 \end{cases}$$

with $\mathcal{L}_g$ spatial, $g = g_R + i g_I$. X such that $\mathscr{F}_S(X)$ is a random measure. U_0 spatial. Solution (Duhamel's formula) having càdlàg behaviour in time:

$$U_t = \mathscr{F}_S^{-1} \left(e^{-tg} \mathscr{F}_S(U_0) + \int_{(0,t]} e^{-(t-u)g} d\mathscr{F}_S(X)(\cdot, u) \right)$$

Asymptotic result: [1] if $g_R \geq \kappa > 0$, existence of unique stationary solution, and the solution of the Cauchy problem converges spatio-temporally to it as $t \to \infty$.

General covariance-SPDE connection

Question: Given a random field Z with covariance C_Z, does it exist a White Noise driven SPDE that Z satisfies? [2]

Theorem 2. *Given Z, there is a linear operator $\mathcal{L}_1$ and a White Noise W_1 such that*

$$Z = \mathcal{L}_1 W_1.$$

In addition, there exists a linear operator $\mathcal{L}_2$ and a White Noise W_2 such that

$$\mathcal{L}_2 Z = W_2$$

if and only if the RKHS of Z is infinite-dimensional.

Auxiliary result: Karhunen-Loève expansion of Z

$$Z = \sum_{n \in \mathbb{N}} Z_n T_n,$$

$(Z_n)_n$ uncorrelated with summable variances, and $(T_n)_{n \in \mathbb{N}}$ base of distributions.

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