On decay-surge models

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Abstract
Continuous space-time decay surge population models are those semi-stochastic ones for which deterministically declining populations, bound to fade away, with bursts or surges of random sizes at random times. We exhibit conditions under which such processes either explode or are transient or recurrent.

Introduction

We consider decay surge population models such that
- the nonlinear decay flow, determines the decay rate of the population.
- the nonlinear rate function, determines the frequencies of the jumps.
- the nonlinear kernel factor determines the amplitudes of the shocks.

Between any two consecutive upward jumps, there is a decrease in the size of the population.

Main Objectives

- To investigate the conditions under which the decay surge processes either explode or are transient at 0 or at infinity, or recurrent.
- To study the hitting times of the process.
- To simulate the corresponding embedded chain and study some of its properties.

Description of the model

Let $\alpha(x)$ be a continuous function on $[0,\infty)$, positive on $(0,\infty)$ or even sometimes on $[0,\infty)$ and $\beta(x)$ a continuous and positive rate function on $(0,\infty)$.

We consider the piecewise deterministic Markov process (PDMP) with state-space $[0,\infty)$ obeying:

$$dX_t = -\alpha(X_{t-})\,dt + \Delta(X_{t-})\int_0^\infty \mathbf{1}_{\{r\le\beta(X_{t-}(x))\}} M(dt,dr), \tag{1}$$

where $M(dt,dr)$ is a Poisson measure on $[0,\infty)\times[0,\infty)$ and X_{t-} is the position of the process just before time t.

At the jump times, the size of the population grows by a random amount $\Delta(X_{t-})>0$ of its current size X_{t-}. We set

$$Y(X_{t-}) := X_{t-} + \Delta(X_{t-})$$

the new state X_t after a jump from X_{t-} and

$$\mathbf{P}(Y(x) > y \mid X_- = x) = k(y)/k(x).$$

The infinitesimal generator of the process $X_t(x)$ is

$$(\mathcal{G}u)(x) = -\alpha(x)u'(x) + \frac{\beta(x)}{k(x)}\int_x^\infty k(y)u'(y)\,dy$$

where u is a smooth function.
In the following, we set $\gamma(x)=\beta(x)/\alpha(x)$ and $\Gamma(x)=\int^x \gamma(y)dy$.

Assumptions

1. **Condition (R):** (i) $\beta(0)>0$ and
   (ii) $K(0,y)=k(y)/k(0)>0$ for some $y>0$ (and in particular $k(0)<\infty$).
2. **Assumption (C)** : $\int^\infty \frac{\gamma(y)}{k(y)}e^{-\Gamma(y)}dy<\infty$.

Results

Proposition 1. *If we suppose an invariant measure (or speed measure) $\pi(dy)$ exists then its explicit form is*

$$\pi(y) = C\frac{k(y)\,e^{\Gamma(y)}}{\alpha(y)}, \tag{2}$$

where C is a positive constant.

Proposition 2. *Suppose $\Gamma(\infty)=\infty$ and that* **Assumption (C)** *does not hold. Suppose also that β is continuous on $[0,\infty)$. Let $V_1(x) = 1+s_2(x)$ and $V_2(x) = 1-c(x)(1-x)$ where $c(x)=\frac{1}{2}+x(c-\frac{1}{2})$ and $c=\gamma(1)e^{-\Gamma(1)}/k(1)$. We set*

$$V(x) = \begin{cases} V_1(x), & \text{if } x\ge 1,\\ V_2(x), & \text{if } x\in[0,1].\end{cases}$$

Then $V\in\mathcal{C}^1$ and $V(x)\ge 1/2$ for all x. V is a norm-like function in the sense of [3], and we have
1. $\mathcal{G}V(x)=0,\ \forall x\ge 1.$
2. $\sup_{x\in[0,1]}|\mathcal{G}V(x)|<\infty.$
As a consequence, $S_\infty=\sup_n S_n=\infty$ almost surely, so that X is non-explosive.

Theorem 1. *Grant* **Condition** (R), *suppose that $\Gamma(\infty)=\infty$ and moreover that $\int^\infty \frac{\gamma(y)}{k(y)}e^{-\Gamma(y)}dy=\infty$. Then the process is recurrent in the sense of Harris and its unique invariant measure is given by π.*

Proposition 3. *Suppose $\Gamma(\infty)=\infty$ and* **Assumption (C)** *holds. Let $0<a<x<b<\infty$ and suppose that $\tau_{x,a}\wedge\tau_{x,b}<\infty$ almost surely. Then*

$$\mathbf{P}(\tau_{x,a}<\tau_{x,b}) = \frac{\int_x^b \frac{\gamma(y)}{k(y)}e^{-\Gamma(y)}dy}{\int_a^b \frac{\gamma(y)}{k(y)}e^{-\Gamma(y)}dy}. \tag{3}$$

where

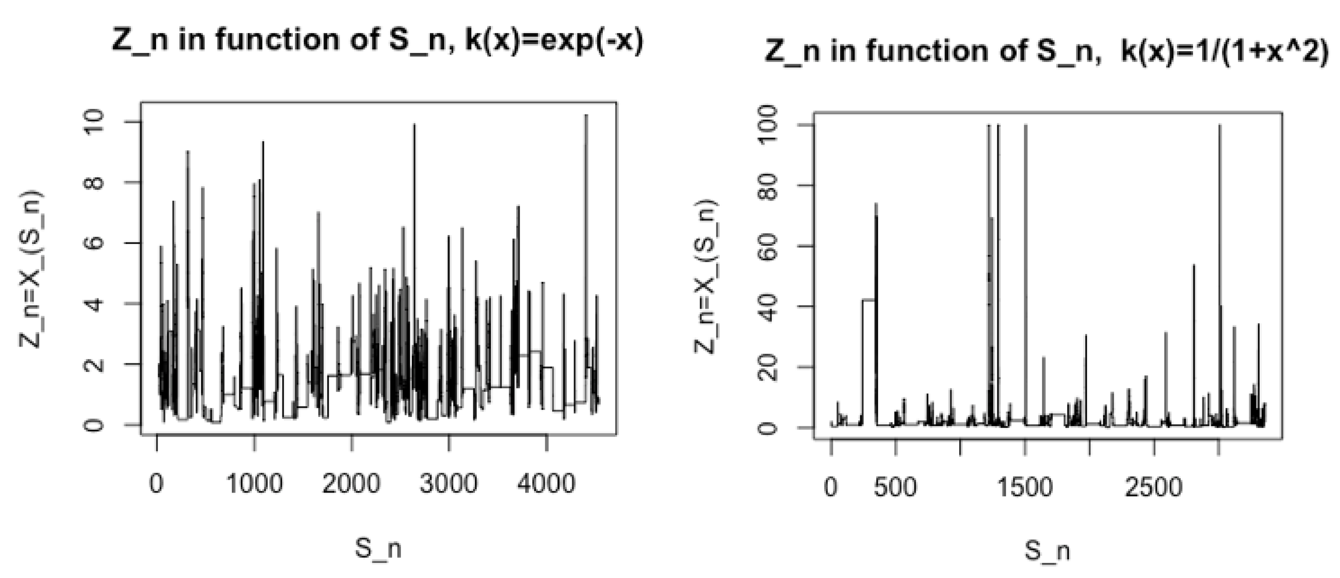
$$\tau_{x,b} = \inf\{t>0 : X_t = b\} = \inf\{t>0 : X_t\le b, X_{t-}>b\}$$

and

$$\tau_{x,a} = \inf\{t>0 : X_t = a\} = \inf\{t>0 = X_t\le a\}.$$

Simulations

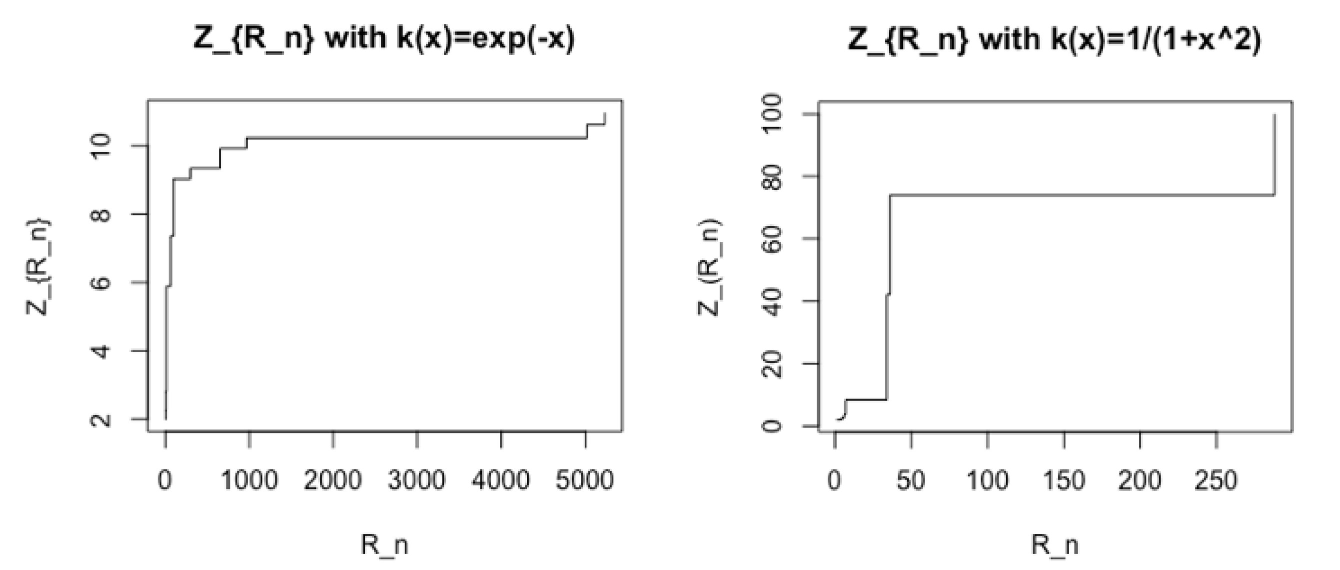
We simulate the embedded chain $Z_n = X_{S_n}$ (where S_n is the jump time of X_n). We take $\alpha(x)=\alpha_1 x^a$ and $\beta(x)=\beta_1 x^b$ with $\alpha_1=1$, $a=2$, $\beta_1=1$ and $b=1$. We also take on one hand $k(x)=e^{-x}$ and the other hand $k(x)=1/(1+x^2)$. The simulation of the jump times S_n then goes through a simple inversion of the conditional distribution function $\mathbf{P}(S_n\le t\mid X_{S_{n-1}}=x, S_{n-1}=s)$. These graphs give the positions Z_n as a function of the jump times S_n.



Successive inter jump times are longer in the first process than in the second one. Jumps reach higher heights in the second process than in the first.

We simulate the upper record times and values sequences of Z_n, namely $R_n = \inf(r\ge 1 : r>R_{n-1}, Z_r > Z_{R_{n-1}})$ and $Z_n^* = Z_{R_n}$.

The following graphs give Z_{R_n} as a function of R_n.



We remark that the gap between two consecutive records decreases over the time whereas the time between two consecutive records becomes longer. In other words, higher is a record, longer it takes to surpass it statistically.

References

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