Stochastic persistence of ecological communities driven by Lévy-noise

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Persistence: preliminaries

The mathematical problem of stochastic coexistence

A simple stochastic LV system of prey-predator type with intraspecific competition driven by Brownian motion (here the flow of biomass is $1 \longrightarrow 2$).

$$
\begin{aligned}
dX_t^{(1)} &= X_t^{(1)}(b_1 - a_{11}X_t^{(1)} - a_{12}X_t^{(2)}) + \sigma_1 X_t^{(1)} dW_t^{(1)} \\
dX_t^{(2)} &= X_t^{(2)}(-b_2 + a_{21}X_t^{(1)} - a_{22}X_t^{(2)}) + \sigma_2 X_t^{(2)} dW_t^{(2)}.
\end{aligned}
\tag{1}
$$

Initial conditions for the process $\mathbf{X} = (X^{(1)}, X^{(2)})$ in $\mathbb{R}_+^n$ = positive orthant ($\mathbb{R}_{++}^n$ = strictly positive orthant).

Biological interpretation:

1. b_i's: per-capita per-unit time birth-death rates.
2. a_{ij}'s: if $i \neq j$, interaction strength (increase-decrease of fitting per unit time per prey-predator encounter). If $i = j$: intraspecific competition (due e.g. to competition for space, soil, light, etc.)
3. σ_i's: amount of the random perturbation due to environmental noise.
4. $\mathbf{W} = (W^{(1)}, W^{(2)})$: standard 2d Wiener process, carrying the **environmental noise**.

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Modeling issue: it is assumed that the stochastic perturbations of the fitting occur in characteristic times much shorter than the time horizon relevant to our observations

In our work we consider communities of n populations indexed by I engaged in LV dynamics driven by continuous and jump-type noise.

We ask: what are the conditions on the parameters such that the whole community $\mathbf{X} = (X^{(i)} : i = 1, 2, \ldots, n)$ **persists**.

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• It can be proved that for initial condition on the strictly positive cone, for every $t > 0$, $X_t^{(1)} > 0, X_t^{(2)} > 0$ a.s. $\Rightarrow$ there is no extinction on finite time. $\Rightarrow$ any notion of stochastic persistence involves a claim on the asymptotic behavior of the laws.

Some different notions of stochastic persistence

Definition

A set of species indexed by $J \subset I$ is said to be **stochastically persistent in probability** if for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \mathbb{R}_{++}^{|J|}$ such that:

$$\liminf_{t\to\infty} \mathbb{P}_{\mathbf{x}} \left((X_t^{(i_1)}, X_s^{(i_2)}, \ldots, X_t^{(i_l)}) \in K_\varepsilon, (i_1, i_2, \ldots, i_l) \in J \right) > 1 - \varepsilon,$$

uniformly on the initial condition $\mathbf{x} \in \mathbb{R}_{++}^{I}$.

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- Previous work has provided condition for these *weak* form of persistence for Lévy-driven ecological models: [BY11], [Mao11], [BMYY12].

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1. when do the laws of $\mathbf{X}$ converges (in an appropriate sense) to a unique invariant probability measure, π?
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2. What is the nature of this invariant measure? Namely: does it support the whole community or just a strict subset of the community?

- If π does not change the boundary of the positive cone, then we say that the community is **strongly stochastic persistent**.

Definition

(see [HN18a]) We say that a $\mathbb{R}^n_{++}$-valued, right-continuous Markov process $(\mathbf{X}_t : t \geq 0)$ is **strongly stochastically persistent** (SSP) if there exists a unique invariant probability measure π supported on $\mathbb{R}^n_{++}$ such that for every $\mathbf{x} \in \mathbb{R}^n_{++}$:

$$d_{TV}(\mathbb{P}_{\mathbf{x}}(\mathbf{X}_t \in \cdot), \pi(\cdot)) \to 0,$$

as $t \to \infty$.

We observe:

1. $\mathbf{X}$ is not empty (because $\delta_{\mathbf{0}}$ is trivially ergodic).
2. For every $J \subseteq \{1, 2\}$, the sets:

$$\mathbb{R}^J_{++} := \{\mathbf{x} \in \mathbb{R}^n_+ : x_i > 0, i \in J, x_i = 0, i \notin J\},$$

   are positively invariant for the dynamics. In general, thus, we can expect to have a lot of invariant probability measures on $\mathbb{R}^n_+$, many concentrated on $\partial\mathbb{R}^n_+$ (the boundary of the orthant, where at least one of the species is extinct).

Crucial insight

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- Pick an invariant probability measure μ such that at least one of the species is extinct under μ (and thus, μ charges $\partial\mathbb{R}^n_+$). Suppose that the process X, when started from strictly positive abundances, gets close to the part of the extinction boundary that is changed by μ. To guarantee SSP we will need that, in this situation, at least one of the endangered species tend to grow rapidly, avoiding thus the extinction fate.

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- Under some regularity assumption, this will amount to ask that, under any invariant probability measure μ that charges the boundary, at least one of the absent species has positive μ-averaged Lyapunov exponent.

Back to first example

How to ensure SSP for this community?

We have many invariant (indeed, ergodic) p.m. for this system.

- $\delta_{0,0}$.
- An ergodic p.m. μ under which species 1 persists and species 2 is extinct (thus, concentrated on $\{x_1 > 0, x_2 = 0\}$).

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- $\delta_{0,0}$.
- An ergodic p.m. μ under which species 1 persists and species 2 is extinct (thus, concentrated on $\{x_1 > 0, x_2 = 0\}$). This p.m. is unique, and corresponds to the product measure $m \otimes \delta_0$, where m is the law of a Gamma-distributed random variable (the law of the stochastic logistic equation).

Apply (formally!) Ito's formula to $\mathbf{x} \mapsto \log(x_1)$ and $\mathbf{x} \mapsto \log(x_2)$ to obtain:

$$\frac{\log(X_t^{(1)})}{t} - \frac{\log(X_0^{(1)})}{t} = \frac{1}{t}\int_0^t \left(b_1 - a_{11}X_s^{(1)} - a_{12}X_s^{(2)} - \frac{1}{2}\sigma_1^2\right) ds + \frac{M_1(t)}{t}$$

$$\frac{\log(X_t^{(2)})}{t} - \frac{\log(X_0^{(2)})}{t} = \frac{1}{t}\int_0^t \left(-b_2 + a_{21}X_s^{(1)} - a_{22}X_s^{(2)} - \frac{1}{2}\sigma_2^2\right) ds + \frac{M_2(t)}{t}$$

where M_1 and M_2 are local martingales. Put:

$$\Xi_1(\mathbf{x}) := b_1 - a_{11}x_1 - a_{12}x_2 - \frac{1}{2}\sigma_1^2$$

$$\Xi_2(\mathbf{x}) := -b_2 + a_{21}x_1 - a_{22}x_2 - \frac{1}{2}\sigma_2^2$$

Intuitively...

1. If X expends a lot of time close to, for example, the support of μ, then for large times the first integrals should be close to $\mu\Xi_1$ and $\mu\Xi_2$.
2. Since under μ the first species persists, $\mu\Xi_1 = 0$. (resident, stable species do not die or grow exponentially fast.)

So, we require:

1. $max_{i=1,2}\delta_{0,0}\Xi_i > 0$ (when both are rare, at least one of the species tends to grow exponentially fast).
2. $\mu\Xi_2 > 0$ (when rare, the predator tends to grow exponentially fast).

From the first condition we obtain:

$$b_1 - \frac{1}{2}\sigma_1^2 > 0$$

and from the second one we get:

$$-b_2 + a_{21}\mathbb{E}_{Z\sim m}(Z) - \frac{1}{2}\sigma_2^2 > 0$$

Since $\mu\Xi_1 = 0$, we have:

$$\mathbb{E}_{Z\sim m}(Z) = \frac{b_1 - \frac{1}{2}\sigma_1^2}{a_{11}}$$

Thus, still intuitively, a sufficient condition for SSP is:

$$\frac{b_1 - \frac{1}{2}\sigma_1^2}{a_{11}} > \frac{b_2 + \frac{1}{2}\sigma_2^2}{a_{21}}$$

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• In the simple example of the prey-predator community, the above intuition is just right ([Ben18], [HN18b]).

• The aim of this work is to show how these results can be extended to communities such as (2).

Our framework, assumption and results

Our SDE framework

$$d\mathbf{X}_t = \mathbf{X}_t \circ \left((\mathbf{B} + \mathbf{A}\mathbf{X}_t)dt + \Sigma d\mathbf{W}_t + \int_{\mathbb{R}^n \setminus \{0\}} \mathbf{L}(\mathbf{X}, \mathbf{z}) \tilde{N}(dt, dz) \right). \quad (2)$$

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($\mathbf{\Sigma}$: constant; $\tilde{N}$: a compensated Poisson random measure of intensity measure $dt\nu(d\mathbf{z})$ for ν a Lévy measure on $\mathbb{R}^n$).

Our SDE framework

$$dX_t = X_t \circ \left((B + AX_t)dt + \Sigma dW_t + \int_{\mathbb{R}^n \setminus \{0\}} L(X, z)\tilde{N}(dt, dz) \right). \quad (2)$$

(Σ: constant; $\tilde{N}$: a compensated Poisson random measure of intensity measure $dt\nu(dz)$ for ν a Lévy measure on $\mathbb{R}^n$).

- **Biological interpretation of the jump part:** Other sources of randomness on the biotic or abiotic factors of the ecosystems whose natural time-scale are much longer than those of the perturbation modeled by the Brownian part: sudden migration of resident species or incoming flow of biomass through ecosystem boundaries (see e.g. the excellent compendium on the topic of animal migration [MG11], specially chapter 9) or ENSO effects on habitat compression and deepening of nutricline (see [OMK+17]).

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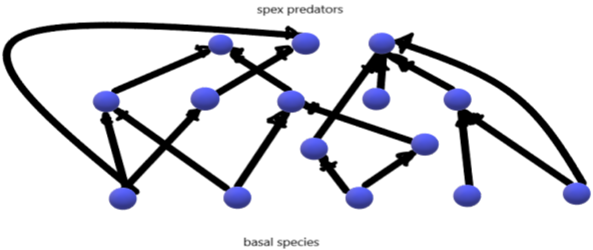
- The signs of **A** determine a directed graph (with an edge from prey to predators).
- We consider **layered communities: food-webs with intraspecific competition with no autocalytic cycles**.



Assumptions

1. The community is layered.
2. (**Assumption 1**): $\int_{\mathbb{R}^n \setminus \{0\}} \|\mathbf{z}\|^2 \nu(d\mathbf{z}) < \infty$.
3. (**Assumption 2**): For every $i = 1, 2, \ldots, n$,$\mathbf{x}, \mathbf{z} \in \mathbb{R}^n$:

$$|L_i(\mathbf{x}, \mathbf{z})| \leq |z_i| \mathbf{1}_{z_i > -1}.$$

4. (**Assumption 3**): There exists a constant C such that:

$$\|\mathbf{x} \circ \mathbf{L}(\mathbf{x}, \mathbf{z})\| \leq C \|\mathbf{z}\|$$

5. (**Assumption 4**): For $\mathbf{x}, \mathbf{y}$ in K compact, $\mathbf{z} \in \mathbb{R}^n$, there exists a constant M_k such that:

$$\|\mathbf{x} \circ \mathbf{L}(\mathbf{x}, \mathbf{z}) - \mathbf{y} \circ \mathbf{L}(\mathbf{y}, \mathbf{z})\| \leq M_k \|\mathbf{x} - \mathbf{y}\| \|\mathbf{z}\|,$$

holds.

Invasibility rates

Just like in our example, define:

$$\Xi_i(\mathbf{x}) := (B_i + (\mathbf{A}\mathbf{x})_i) - \frac{1}{2}\sigma_i^2 - \int_{\mathbb{R}^n_+} L_i(\mathbf{x}, \mathbf{z}) - \ln(1 + L_i(\mathbf{x}, \mathbf{z}))\nu(d\mathbf{z}).$$

More technical assumptions

1. Observe that any layered community has the following property: there exists $\mathbf{c} \in \mathbb{R}^n_{++}$ such that whenever $i \to j$:

$$c_i a_{ij} \geq c_j a_{ji}.$$

2. It is proved that the conditions:

$$\text{For every } \mu \in \mathcal{P}_{erg}(\partial \mathbb{R}^n_+) : \max_i \mu \Xi_i > 0$$

and

$$\text{There exists } \boldsymbol{p} := (p_1, \ldots, p_n) \in \mathbb{R}^n_{++} \text{ such that:}$$
$$\inf_{\mu \in \mathcal{P}_{erg}(\partial \mathbb{R}^n_+)} \sum_i p_i \mu \Xi_i := \rho > 0, \tag{3}$$

are equivalent ([SBA11]).

Crucial assumption

(**Assumption 5**): The condition (3) holds.

log-Lyapunov functions

Define for $\mathbf{x} \in \mathbb{R}^n_{++}$

$$\hat{V}(\mathbf{x}) = \frac{1 + \mathbf{c}^T \mathbf{x}}{\prod_{i=1}^n x_j^{p_j}},$$

and for $\alpha > 0$ define:

$$\tilde{W}(\mathbf{x}) := \hat{V}^{\alpha}(\mathbf{x}).$$

(the c_i's are those ensured by the layered hypothesis; the p_i's are *small* and satisfy the condition (3); ; $\alpha > 0$ will be fixed appropriately).

Observe that for p_i small (namely, $\sum_i p_i < 1$), the function $\hat{V}$ satisfy:

$$\lim_{\|\mathbf{x}\|\to\infty} \hat{V}(\mathbf{x}) = \liminf_{\mathbf{x}:\min_i x_i \to 0} \hat{V}(\mathbf{x}) = +\infty$$

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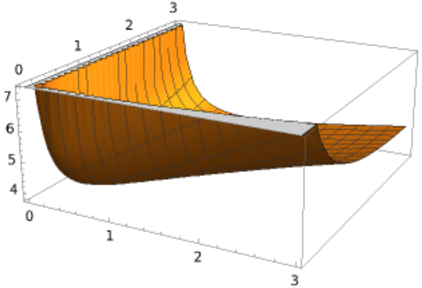


Further assumption

For $\mathbf{x} \in \mathbb{R}^n_{++}$, set:

$$I(\mathbf{x}) := \int_{\mathbb{R}^n} \Big\{ \tilde{W}((x_i(1 + L_i(\mathbf{x}, \mathbf{z}))_i) - \tilde{W}(\mathbf{x}) \\ -\alpha \tilde{W}(\mathbf{x}) \sum_{i=1}^{n} \Big(L_i(\mathbf{x}, \mathbf{z})(\frac{c_i x_i}{1 + \mathbf{c}^T \mathbf{x}} - p_i) \Big) \Big\} \nu(d\mathbf{z}).$$

Two further technical assumptions

Assumption 6:

$$I(\mathbf{x}) \leq C\tilde{W}(\mathbf{x})$$

for some positive constant C. Furthermore, for some small positive α_0, the function $\mathbf{z} \mapsto \exp(\alpha_0 \|\mathbf{z}\|)$ is ν-integrable.

Main theorem

Theorem

Suppose Assumptions 1 *through* 6 *hold. Then* $(\mathbf{X}_t : t \geq 0)$ *is a* $\mathbb{R}_+^n$*-valued, right-continuous,* $\mathcal{C}_b$*-Feller Markov process with the SSP property.*

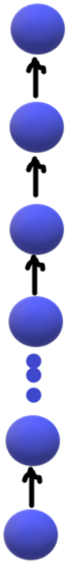
Remark

Define:

$$\text{Safe}(\varepsilon, R) := \{\mathbf{x} \in \mathbb{R}^n_{++} : \min x_i > \varepsilon, \|\mathbf{x}\| \leq R\},$$

and let $\mathcal{I}(\mathbf{x})$ be a positive Lipschitz function, bounded by 1, that approximates $\mathbf{1}_{\text{Safe}(\varepsilon,R)}(\mathbf{x})$ from below.
Assume that the community is layered and $\nu(d\mathbf{z}) = \lambda\mu(d\mathbf{z})$, with μ the law of a $\mathbb{R}^n$-valued random variable with compact support contained in $\{z_i > -\gamma, i = 1, \ldots, n\}$, where $\gamma \in (0, 1)$. Assume also that $L_i(\mathbf{x}, \mathbf{z}) = \beta\mathcal{I}(\mathbf{x})z_i$, where $0 < \beta < 1$, $\varepsilon > 0$ and $R > 0$.
On this setting, the hypotheses 1, 2, 3, 4 and 6 required by Theorem 3 are satisfied.

Example: Lévy-driven, Lotka-Volterra food-chains



Parameters

For $i = 1, \dots, n$, let $(N_t^{(i)} : t \geq 0)$ be a family of independent compound Poisson processes on $\mathbb{R}$ with intensity $\tilde{\lambda}_i > 0$ and jump increment distributed according to a law m_i with support contained in $\{z > -1\}$. As before, let $\tilde{N}_i$ be the compensated process. Let $\beta \in (0, 1)$ and $0 < \varepsilon_i < M_i$, $i = 1, \dots, n$ be some constants. Let $\ell_i : \mathbb{R}_+ \to [0, 1]$ be a Lipschitz function that equals 1 on $[\varepsilon_i, M_i]$ and vanishing outside the interval $[\varepsilon_i/2, 2M_i]$. Set $L_i(\mathbf{x}, \mathbf{z}) = \ell_i(x_i)\beta z_i$.

The food-chain equations

$$
\begin{aligned}
dX_t^{(1)} &= X_t^{(1)} \Bigg((b_1 - a_{11} X_t^{(1)} - a_{12} X_t^{(2)}) dt + \sigma_1 dW_t^{(1)} \\
&\qquad + \int_{\{z_1 > -1\}} L_1(\mathbf{X}_t, \mathbf{z}) \tilde{N}_1(dz_1, dt) \Bigg), \\
dX_t^{(i)} &= X_t^{(i)} \Bigg((-b_i + a_{i,i-1} X_t^{(i-1)} - a_{i,i} X_t^{(i)} - a_{i.i+1} X_t^{(i+1)}) dt + \sigma_i dW_t^{(i)} \\
&\qquad + \int_{\{z_i > -1\}} L_i(\mathbf{X}_t, \mathbf{z}) \tilde{N}_i(dz_i, dt) \Bigg), \quad i = 2, \dots, n-1, \\
dX_t^{(n)} &= X_t^{(n)} \Bigg((-b_n + a_{n,n-1} X_t^{(n-1)} - a_{nn} X_t^{(n)}) dt + \sigma_n dW_t^{(n)} \\
&\qquad + \int_{\{z_n > -1\}} L_n(\mathbf{X}_t, \mathbf{z}) \tilde{N}_n(dz_n, dt) \Bigg).
\end{aligned} \tag{4}
$$

SSP for Lévy-driven LV food-chains

For $n \geq 2$, set:

$$d_1 = b_1 - \frac{1}{2}\sigma_1^2, \quad d_i = b_i + \frac{1}{2}\sigma_i^2; \quad i = 2\ldots, n$$

$$\Delta_i = \mathbb{E}_{Z\sim m_i}(\beta Z - \ln(1+\beta Z)) \geq 0; \quad \lambda_i = \tilde{\lambda}_i \Delta_i$$

$$r_1 := d_1 - \lambda_1, \quad r_i := d_i + \lambda_i, i = 2, \ldots, n;$$

$$\mathbf{r}_m = (-r_1, , r_2, \ldots, r_m), 1 \leq m \leq n.$$

SSP for Lévy-driven LV food-chains

Theorem

For $2 \leq m \leq n$, *let* $\mathbf{A}_m$ *be the leading m-dimensional submatrix of* $\mathbf{A}$. *Consider the linear systems:*

$$\mathbf{A}_m s = \mathbf{r}_m, \quad 2 \leq m \leq n$$

and let $s^{(*,m)} \in \mathbb{R}^m$ *be the unique solution of the m-th system . Assume that:*

$$a_{n,n-1} s_{n-1}^{(*,n-1)} > r_n. \tag{5}$$

holds. Then the SDE (4) *is SSP.*

Remarks:

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1. The above result extends a Theorem of Hening and Nguyen ([HN18c]), which in turn extends a result of Gard ([GH79]) on deterministic LV food-chains.

Appendix: sketch of the proof

Steps toward the proof

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- To prove that there exists an **embedded chain** with tight laws in $\mathbb{R}^n_{++}$ (uniformly for initial conditions on compact sets). A sufficient condition is:

Proposition

For some continuous log-Lyapunov function $\tilde{V}$, for some $T^* > 0$, $m \in (0,1)$ and $C > 0$, the inequality:

$$\mathbb{E}_{\mathbf{x}}(\tilde{V}(\mathbf{X}_{T^*})) \leq m\tilde{V}(\mathbf{x}) + C,$$

holds for every $\mathbf{x} \in \mathbb{R}^n_{++}$

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holds for every $\mathbf{x} \in \mathbb{R}^n_{++}$

- This is the hard part.

Steps toward the proof

We prove:

Lemma

For every $T > 0$ *and* $\mathbf{X}_0 \in \mathbb{R}^n_{++}$, *under* $\mathbb{P}_{\mathbf{X}_0}$ *the Markov chain* $\mathbf{X}^T := (\mathbf{X}_{mT} : m \geq 0)$ *is irreducible and aperiodic. Furthermore, every compact set is a petite set for the chain.*

By Lemma 5 and Theorem 6.3 of [MT92], there exists a probability measure π such that:

$$\lim_{n\to\infty} \|\mathbb{P}_{\mathbf{x}}(\mathbf{X}_{nT^*} \in \cdot) - \pi(\cdot)\|_{TV} = 0,$$

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and the convergence is geometric. Then, $\mathbf{X}^{T*}$ is positive Harris recurrent, and thus this chain has hitting times (of open sets) with finite expectation; this in turn also implies that $\mathbf{X}$ is positive Harris recurrent (see theorem 1 of [KM94]), and thus has a unique invariant probability measure in $\mathbb{R}^n_{++}$. Of course, this probability measure is just π.

M. Benaim.
Stochastic persistence.
Retrieved december 25, 2020 from: https://arxiv.org/abs/1806.08450, 2018.

J. Bao, X. Mao, G. Yin, and C. Yuan.
Competitive lotka–volterra population dynamics with jumps.
Nonlinear Analysis, 74:6601–6616, 2012.

J. Bao and C. Yuan.
Stochastic population dynamics driven by lévy noise.
J. Math. Anal. Appl., 391:363–375, 2011.

T.C. Gard and T.G. Hallam.
Persistence in food-webs. i. lotka-volterra food-chains.
Bull. Math. Biol., 41(6):877–891, 1979.

A. Hening and D. Nguyen.
Coexistence and extinction for stochastic kolmogorov systems.
Ann.Appl.Prob., 28(3):1893–1942, 2018.

A. Hening and D. Nguyen.

Stochastic lotka-volterra food chain.
Journal of Mathematical Biology, 77(1):135–163, 2018.

A. Hening and D.H. Nguyen.
Persistence in stochastic lotka-volterra food chains with intraspecific competition.
Bull. Math. Biol., 80:2527–2560, 2018.

H. Kaspi and A. Mandelbaum.
On harris recurrence in continuous time.
Mathematics of operation research, 19(1), 1994.

X. Mao.
Stochastic Differential equations with applications.
Woodhead Publishing, Cambridge, UK, 2011.

E. Milner-Gulland.
Animal Migration: A Synthesis.
Oxford University Press, 2011.

S. Meyn and R. Tweedie.
Stability of markovian processes i: Criteria for discrete-time chains.

Adv. Appl. Prob-, 24(3):542–574, 1992.

M. Ohman, N. Mantua, J. Keister, M. García-Reyes, and S. McClatchie.
Enso impacts on ecosystem indicators in the california current system.
Retrieved January 23th, 2021 from https://www.us-ocb.org/enso-impacts-on-ecosystem-indicators-in-the-california-current-system/, 2017.

S. Schreiber, M. Benaïm, and K. Atchadé.
Persistence in fluctuating environments.
J. Math. Biol., (62):655—-683, 2011.