Scaling and Taylor's law for heavy tailed observations

with J. Cohen and R. Davis

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- Taylor's law: for some positive constants a, b:

$$S_n^2 \approx a(\bar{X}_n)^b \text{ for moderately large } n.$$

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- We focus on Taylor's law and related scaling resulting from heavy tails.

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- X is in RV(α) means $\bar{F}(x) = x^{-\alpha}L(x)$, L slowly varying.

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- S_α: a totally skewed to the right α-stable random variable.

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- $\left(S_\alpha, S_{\alpha/2}\right)$ marginally stable, jointly infinitly divisible.

$$S_\alpha = \begin{cases} \sum_{j=1}^{\infty} \left(\Gamma_j^{-1/\alpha} - E\left[\Gamma_j^{-1/\alpha} \mathbf{1} \left(\Gamma_j^{-1/\alpha} \le 1 \right) \right] \right) & \text{if } 1 \le \alpha < 2 \\ \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} & \text{if } 0 < \alpha < 1 \end{cases} .$$

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- $\Gamma_1, \Gamma_2, \ldots$: standard Poisson arrivals.

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- $a_n = n^{1/\alpha} L_1(n)$, L_1 slowly varying.

$$\big(L_1(n)\big)^{b-2} W_n = \frac{a_n^{-2} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2}{\big(a_n^{-1} \sum_{i=1}^{n} X_i\big)^{b}}$$

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an approximate Taylor's law.

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  - $Z_0, Z_1, Z_2, \dots$ iid standard normals, $0 \leq \rho < 1$, $0 < \alpha < 1$.

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- A particular model for stationary, dependent, heavy tailed observations:
  - $Z_0, Z_1, Z_2, \ldots$ iid standard normals, $0 \le \rho < 1$, $0 < \alpha < 1$.

$$X_i = \frac{1}{\left|\rho^{1/2} Z_0 + (1-\rho)^{1/2} Z_i\right|^{1/\alpha}}, \; i = 1, 2, \ldots.$$

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$$\frac{S_n^2}{\bar{X}_n^b} \Rightarrow \frac{S_{\alpha/2}}{S_\alpha^b}\left(\frac{2}{\pi}\right)^{\alpha/(2(1-\alpha))}\left[(1-\rho)\exp\left\{\frac{\rho}{1-\rho}Z_0^2\right\}\right]^{1/(2(1-\alpha))}.$$

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$$\hat{\kappa}_n = \frac{n^{-1}\sum_{i=1}^n \left(X_i - \bar{X}_n\right)^4}{\left(n^{-1}\sum_{i=1}^n \left(X_i - \bar{X}_n\right)^2\right)^2}.$$

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- If $X_1, X_2, \ldots$ iid, $EX^4 < \infty$, then $\hat{\kappa}_n \to \kappa$ a.s.

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- What happens when variances are infinite?

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$\big(S_{\alpha/2,12}, S_{\alpha/2,1}, S_{\alpha/2,2}\big)$: a 3-dimensional $\alpha/2$-stable random vector

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- $\big(S_{1,j}, S_{2,j}\big)$, $j = 1, 2, \ldots$: iid random vectors on $\{s_1^2 + s_2^2 = 1,\ s_1, s_2 \geq 0\}$,
independent of the Poisson arrivals Γ_j, $j \geq 1$.

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$$m(A) = \nu \left\{ (x_1, x_2):\ x_1^2 + x_2^2 > 1,\ (x_1, x_2)/\sqrt{x_1^2 + x_2^2} \in A \right\}$$

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  equivalently, X and Y are asymptotically independent.

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- **Example**: (Z_1, Z_2) is bivariate normal, zero means, unit variances, correlation $\rho \in (-1, 1)$.

  Set $X = |Z_1|^{-c}$, $Y = |Z_2|^{-c}$, some $c > 1$.

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$$R_1 \stackrel{\mathrm{d}}{=} \frac{\sum_{j=1}^{\infty} \Gamma_j^{-c}}{\sqrt{\sum_{j=1}^{\infty} \Gamma_j^{-2c}}}.$$

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- The least squares estimator of η:

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a much faster rate of convergence.

Taylor's law, with correlations, $\alpha = 0.1$