Random convex hulls & other extreme-value problems

Julien Randon-Furling

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&
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EcoDep Conference, CY University, Cergy, 15 Sept 2021

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WARNING: I come from Statistical Physics, so *Pardon my ~~French~~ Maths.*

Convex Hull of m Gaussian walks

Results

Proofs – main ingredient(s)

Brownian (and Lévy) convex hulls

2-dimensional case: counting edges

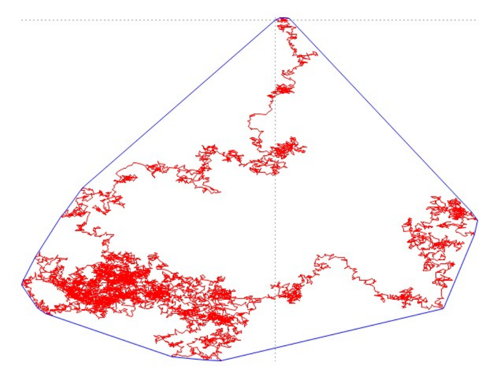
d-dimensional case: counting facets

Some other extreme-value problems

A first-passage time problem for the Brownian supremum

Convex Hull of m Gaussian walks

Convex hull of a single random walk

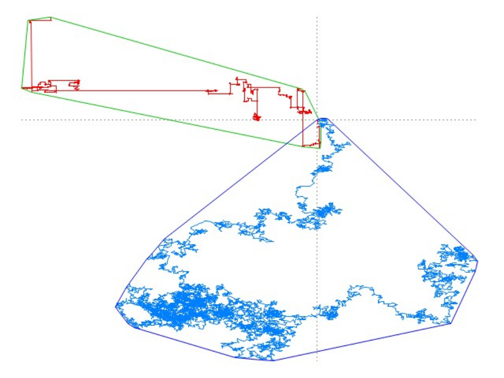


- In dimension 2, for a random walk with n steps:

$$\mathbb{E}\left[\left|\mathcal{F}\left(\mathcal{C}_2\right)\right|\right] = 2\sum_{k=1}^{n}\frac{1}{k}$$

Baxter (1961) ***The Annals of Mathematical Statistics*, 32(3), 901-904.**
Spitzer & Widom (1961) ***Proceedings of the American Mathematical Society*, 12(3), 506-509.**

Convex hull of a single random walk



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- whatever the (symmetric, continuous) distribution of the jumps

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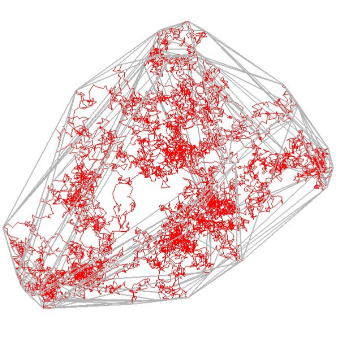


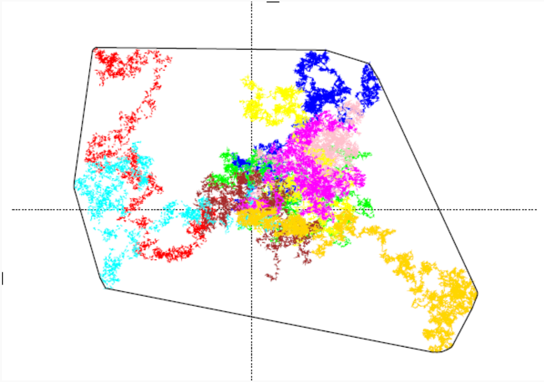
- In dimension d, for a random walk with n steps:

$$\mathbb{E}\left[\left|\mathcal{F}\left(\mathcal{C}_d\right)\right|\right] = 2 \sum_{\substack{j_1+\cdots+j_{d-1}\leq n \\ j_1,\ldots,j_{d-1}\geq 1}} \frac{1}{j_1 \cdot j_2 \cdot \ \ldots \ \cdot j_{d-1}},$$

- whatever the (symmetric, continuous) distribution of the jumps

Barndorff-Nielsen & Baxter (1963) *Transactions of the American Mathematical Society*, 108(2), 313-325.
Vysotsky & Zaporozhets (2018) *Transactions of the American Mathematical Society*, 370(11), 7985-8012.
Kabluchko, Vysotsky & Zaporozhets (2017) *Advances in Mathematics*, 320, 595-629.
R-F & Wespi (2017) *Physical Review E*, 95(3), 032129.

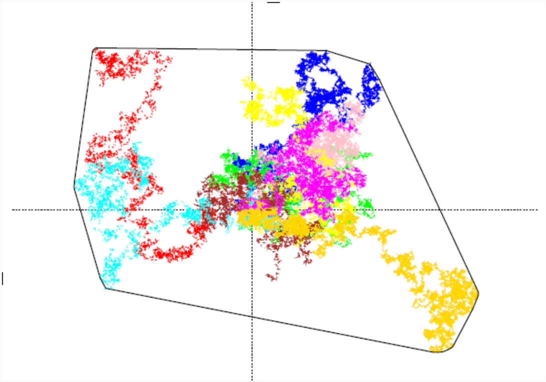
Convex hull of m random walks



- What about the global convex hull of multiple (independent) random walks?

R-F, Majumdar, & Comtet (2009) *Physical Review Letters*, 103(14), 140602.
R-F (2012) *Journal of Physics A: Mathematical and Theoretical*, 46(1), 015004.
Dewenter, Claussen, Hartmann, & Majumdar (2016) *Physical Review E*, 94(5), 052120.

Convex hull of m random walks



- Expected number of edges on the boundary of the global convex hull?

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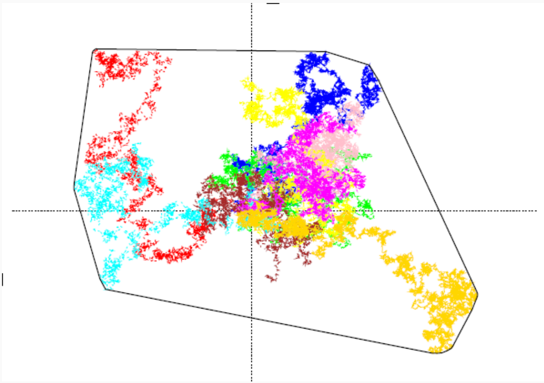
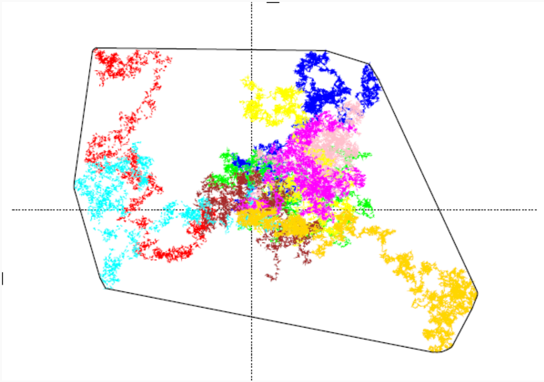
Convex hull of m random walks



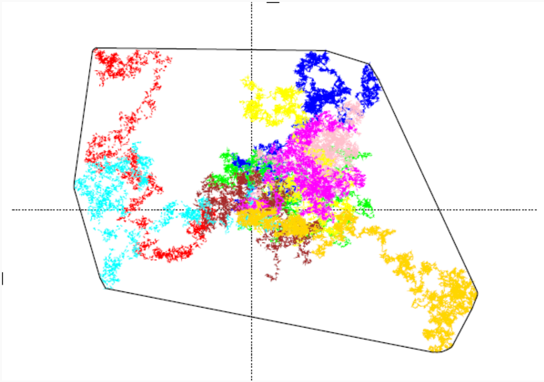
- Expected number of edges on the boundary of the global convex hull?
- More generally, in dimension d: expected number of faces?

Convex hull of m random walks



- Expected number of edges on the boundary of the global convex hull?
- More generally, in dimension d: expected number of faces?
- Not distribution-free: *eg* m single-step random walks $\longleftrightarrow$ m iid points (with 0)

Convex hull of m random walks



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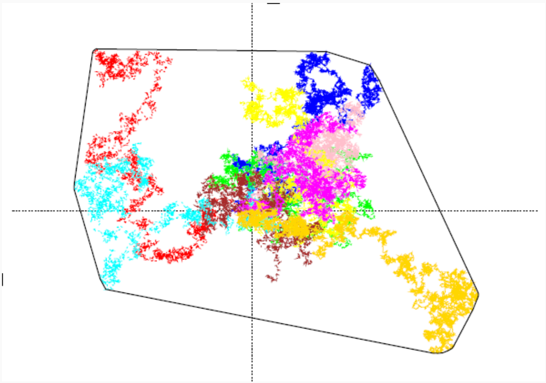
Efron (1965) *Biometrika*, 52(3/4), 331-343.

Rényi & Sulanke (1963, 1964) *Probability Theory and Related Fields*, 2(1), 75-84 & 3(2), 138-147.

Kabluchko & Zaporozhets (2018) *Transactions of the American Mathematical Society*, 372(3), 1709-1733.

Convex hull of m Gaussian random walks



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Efron (1965) ***Biometrika*****, 52(3/4), 331-343.**

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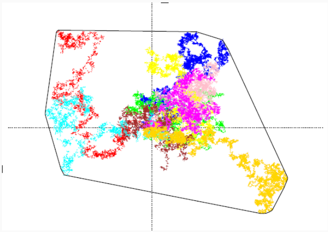
Setting

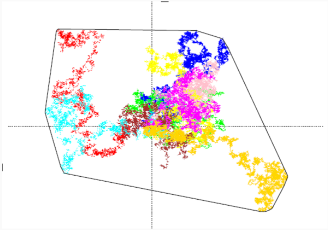


For $m, n_1, \dots, n_m \in \mathbb{N}$, let

$$X_{\mathbf{1}}^{(1)}, \dots, X_{n_{\mathbf{1}}}^{(1)}, \quad \dots, \quad X_{\mathbf{1}}^{(m)}, \dots, X_{n_m}^{(m)}$$

be independent d-dimensional standard Gaussian vectors.

Setting



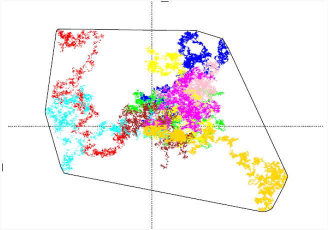
For $m, n_1, \dots, n_m \in \mathbb{N}$, let

$$X_1^{(1)}, \dots, X_{n_1}^{(1)}, \quad \dots, \quad X_1^{(m)}, \dots, X_{n_m}^{(m)}$$

be independent d-dimensional standard Gaussian vectors.
Define the corresponding random walks

$$S_i^{(l)} = X_1^{(l)} + \dots + X_i^{(l)}, \quad 1 \le l \le m, \; 1 \le i \le n_l,$$

Setting



Define the corresponding random walks

$$S_i^{(l)} = X_1^{(l)} + \cdots + X_i^{(l)}, \quad 1 \leq l \leq m,\ 1 \leq i \leq n_l,$$

and a degenerate random walk $\left(S_i^{(0)}\right)_{i=1}^{1}$, with $S_1^{(0)} \equiv 0$.

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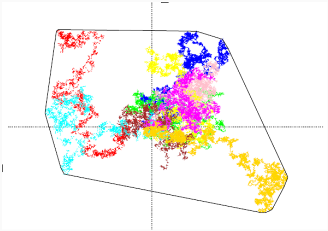

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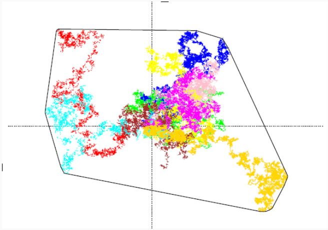
The global convex hull is

$$\mathcal{C}_d = \mathrm{conv}\left\{S_1^{(0)}, S_1^{(1)}, \ldots, S_{n_1}^{(1)}, \quad \ldots, \quad S_1^{(m)}, \ldots, S_{n_m}^{(m)}\right\}.$$

Setting



- With probability one, $\mathcal{C}_d$ is a convex polytope with boundary of the form

$$\partial \mathcal{C}_d = \bigcup_{F \in \mathcal{F}(\mathcal{C}_d)} F,$$

where $\mathcal{F}(\mathcal{C}_d)$ stands for the set of $(d-1)$-dimensional faces of $\mathcal{C}_d$.

- Each face is a (d-1)-dimensional simplex almost surely.

Setting

- Let $k_0, \ldots, k_m$ be integers s.t. $k_0 + \cdots + k_m = d$ and let $i_1^{(l)} < \cdots < i_{k_l}^{(l)} \leq n_l$ be indices, for those $l \in \{0, \ldots, m\}$ s.t. $k_l > 0$.

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- write S_d for the d-tuple

$$\mathrm{S}_d := \left(S_{k_0}^{(0)}, S_{i_1^{(1)}}^{(1)}, \dots, S_{i_{k_1}^{(1)}}^{(1)}, \dots, S_{i_1^{(m)}}^{(m)}, \dots, S_{i_{k_m}^{(m)}}^{(m)} \right)$$

with the convention that $\left\{ S_{i_1}^{(l)}, \dots, S_{i_{k_l}}^{(l)} \right\} := \emptyset$ whenever $k_l = 0$.

Setting

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- also write

$$\operatorname{conv} \mathsf{S}_d := \operatorname{conv} \left\{ S^{(0)}_{k_{\mathbf{0}}}, S^{(\mathbf{1})}_{i^{(\mathbf{1})}_{\mathbf{1}}}, \ldots, S^{(\mathbf{1})}_{i^{(\mathbf{1})}_{k_{\mathbf{1}}}}, \ldots, S^{(m)}_{i^{(m)}_{\mathbf{1}}}, \ldots, S^{(m)}_{i^{(m)}_{k_m}} \right\}.$$

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Note that

- $\operatorname{conv} \mathsf{S}_d$ may be or not be a face of $\mathcal{C}_d$,
- every face can be represented as some $\operatorname{conv} \mathsf{S}_d$.

Setting

- write S_d for the d-tuple

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- $\operatorname{conv} \mathsf{S}_d$ may be or not be a face of $\mathcal{C}_d$,
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- Hence the crucial, albeit elementary, relation:

$$\sum_{F \in \mathcal{F}(\mathcal{C}_d)} g(F) = \sum_{\substack{k_0 + \cdots + k_m = d \\ 0 \le k_l \le n_l,\ l = 0, \ldots, m}} \sum_{\substack{1 \le i^{(l)}_1 < \cdots < i^{(l)}_{k_l} \le n_l \\ l = 0, \ldots, m:\ k_l > 0}} g(\mathsf{S}_d) \mathbb{I}_{\{\mathsf{S}_d \in \mathcal{F}(\mathcal{C}_d)\}} \text{ a.s.,}$$

with $g : \mathbb{R}^d \to \mathbb{R}^1$ an arbitrary, symmetric, non-negative, measurable function.

Setting

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  with $g : \mathbb{R}^d \to \mathbb{R}^1$ an arbitrary, symmetric, non-negative, measurable function.

- relevant choices of g will yield our results

Setting – some more notations

- Unconditional and conditional Gaussian persistence probabilities:
  $p_n(r) := \mathbb{P}\big[\sum_{i=1}^{k} N_i \le r,\ k = 1, \ldots, n\big]$,
  $q_n(r) := \mathbb{P}\big[\sum_{i=1}^{k} N_i \le r, k = 1, \ldots, n \,|\, \sum_{i=1}^{n} N_i = r\big]$,
  where $N_1, \ldots, N_n \in \mathbb{R}^1$ are independent standard Gaussian random variables.
  By symmetry of the distribution:
  $q_n(r) := \mathbb{P}\big[\sum_{i=1}^{k} N_i \ge 0, k = 1, \ldots, n \,|\, \sum_{i=1}^{n} N_i = r\big]$.

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  Note that

$$p_1(r) = \Phi(r) \text{ and } q_1(r) = 1\ \forall r \geq 0,$$
$$p_n(0) = \frac{(2n-1)!!}{(2n)!!} \text{ and } q_n(0) = \frac{1}{n},$$

  where $\Phi(r)$ is the cdf of the standard Gaussian law, and the 3rd & 4th points were established by Sparre Andersen.

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- P_d is the orthogonal projection onto the first $d-1$ coordinates.
- $|\cdot|$ denotes volume or cardinality. $\kappa_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$ is the volume of the d-dimensional unit ball.
- $\mathcal{Q}$ is a matrix chosen uniformly from the orthogonal group $O(d)$, independently with the random walks.

A general formula

Theorem

For $g : \mathbb{R}^d \to \mathbb{R}^1$ *a bounded measurable function, symmetric and invariant with respect to translations,*

$$\mathbb{E}\left[g(\mathsf{S}_d)\mathbb{I}_{\{\mathrm{conv}\,\mathsf{S}_d \in \mathcal{F}(\mathcal{C}_d)\}}\right] =$$

$$d!\,\kappa_d\,(2\pi)^{-d/2} \times \mathbb{E}[g(\mathcal{Q}\mathsf{T}_{d-1}) \cdot |\mathrm{conv}\,\mathsf{T}_{d-1}|]$$

$$\times \prod_{l\,:\,k_l \neq 0} \left[\frac{(2(n_l - i^{(l)}_{k_l}) - 1)!!}{(2(n_l - i^{(l)}_{k_l}))!!} \left(i^{(l)}_1 (i^{(l)}_2 - i^{(l)}_1)^3 \dots (i^{(l)}_{k_l} - i^{(l)}_{k_l - 1})^3 \right)^{-1/2} \right]$$

$$\times \left\{ \mathbb{I}_{\{k_0=0\}} \int_0^\infty \left[\prod_{\substack{l\,:\,k_l=0 \\ l\neq 0}} p_{n_l}(r) \right] \left[\prod_{l\,:\,k_l\neq 0} q_{i^{(l)}_1}(r) \right] \exp\left(-\frac{r^2}{2} \sum_{l\,:\,k_l\neq 0} \frac{1}{i^{(l)}_1} \right) \mathrm{d}r \right.$$

$$\left. + \sqrt{2\pi}\, \mathbb{I}_{\{k_0=1\}} \prod_{l\,:\,k_l=0} \frac{(2n_l-1)!!}{(2n_l)!!} \prod_{\substack{l\,:\,k_l\neq 0 \\ l\neq 0}} \frac{1}{i^{(l)}_1} \right\}.$$

where $\mathsf{T}_{d-1} \sim P_d \mathsf{S}_d$ is a (d-1)-simplex defined from the same indices as S_d.

Face probability

- Applying the previous theorem to $g \equiv 1$ leads to:

Theorem

$$\mathbb{P}\left[\operatorname{conv}\mathsf{S}_d \in \mathcal{F}(\mathcal{C}_d)\right] = d!\,\kappa_d\,(2\pi)^{-d/2} \times \mathbb{E}|\operatorname{conv}\mathsf{T}_{d-1}|$$
$$\times \prod_{l\,:\,k_l\neq 0}\left[\frac{(2(n_l - i_{k_l}^{(l)})-1)!!}{(2(n_l - i_{k_l}^{(l)}))!!}\left(i_1^{(l)}\left(i_2^{(l)} - i_1^{(l)}\right)^3 \ldots \left(i_{k_l}^{(l)} - i_{k_l-1}^{(l)}\right)^3\right)^{-1/2}\right]$$
$$\times \left\{\mathbb{I}_{\{k_0=0\}}\int_0^\infty \left[\prod_{\substack{l\,:\,k_l=0\\ l\neq 0}} p_{n_l}(r)\right]\left[\prod_{l\,:\,k_l\neq 0} q_{i_1^{(l)}}(r)\right]\exp\left(-\frac{r^2}{2}\sum_{l\,:\,k_l\neq 0}\frac{1}{i_1^{(l)}}\right)\mathrm{d}r\right.$$
$$\left. + \sqrt{2\pi}\,\mathbb{I}_{\{k_0=1\}}\prod_{l\,:\,k_l=0}\frac{(2n_l-1)!!}{(2n_l)!!}\prod_{\substack{l\,:\,k_l\neq 0\\ l\neq 0}}\frac{1}{i_1^{(l)}}\right\}.$$

Expected number of faces

- Summing the previous formula over all choices of k_l's and i_j's leads to:

Theorem

$$\mathbb{E}\,|\mathcal{F}(\mathcal{C}_d)| \;=\; d!\,\kappa_d\,(2\pi)^{-d/2} \sum_{\substack{k_0,\dots,k_m\geq 0\\ k_0\leq n_0,\dots,k_m\leq n_m\\ k_0+\dots+k_m=d}} \;\sum_{\substack{1\leq i_1^{(l)}<\dots<i_{k_l}^{(l)}\leq n_l\\ l=0,\dots,m\,:\,k_l>0}} \mathbb{E}|\mathrm{conv}\,\mathsf{T}_{d-1}|$$

$$\times \prod_{l\,:\,k_l\neq 0}\left[\frac{(2(n_l-i_{k_l}^{(l)})-1)!!}{(2(n_l-i_{k_l}^{(l)}))!!}\left(i_1^{(l)}\left(i_2^{(l)}-i_1^{(l)}\right)^3\dots\left(i_{k_l}^{(l)}-i_{k_l-1}^{(l)}\right)^3\right)^{-1/2}\right]$$

$$\times\Bigg\{\mathbb{I}_{\{k_0=0\}}\int_0^\infty\Bigg[\prod_{\substack{l\,:\,k_l=0\\ l\neq 0}} p_{n_l}(r)\Bigg]\Bigg[\prod_{l\,:\,k_l\neq 0} q_{i_1^{(l)}}(r)\Bigg]\exp\Bigg(-\frac{r^2}{2}\sum_{l\,:\,k_l\neq 0}\frac{1}{i_1^{(l)}}\Bigg)\mathrm{d}r$$

$$+\sqrt{2\pi}\,\mathbb{I}_{\{k_0=1\}}\prod_{l\,:\,k_l=0}\frac{(2n_l-1)!!}{(2n_l)!!}\prod_{\substack{l\,:\,k_l\neq 0\\ l\neq 0}}\frac{1}{i_1^{(l)}}\Bigg\}.$$

Expected surface area of the boundary

- Applying the main theorem to $g(\mathsf{S}_d) = |\mathrm{conv}\,\mathsf{S}_d|$, we obtain the expected surface area (i.e. $(d-1)$-dimensional content) of the boundary of the convex hull, $\partial\mathcal{C}_d$:

Theorem

$$\begin{aligned}
\mathbb{E}\,|\partial\mathcal{C}_d| \;=\; & d!\,\kappa_d\,(2\pi)^{-d/2} \sum_{\substack{k_0,\dots,k_m\geq 0\\ k_0\leq n_0,\dots,k_m\leq n_m\\ k_0+\dots+k_m=d}} \;\sum_{\substack{1\leq i_1^{(l)}<\dots<i_{k_l}^{(l)}\leq n_l\\ l=0,\dots,m\,:\,k_l>0}} \mathbb{E}|\mathrm{conv}\,\mathsf{T}_{d-1}|^2 \\
& \times \prod_{l\,:\,k_l\neq 0}\left[\frac{(2(n_l-i_{k_l}^{(l)})-1)!!}{(2(n_l-i_{k_l}^{(l)}))!!}\Big(i_1^{(l)}\big(i_2^{(l)}-i_1^{(l)}\big)^3\dots\big(i_{k_l}^{(l)}-i_{k_l-1}^{(l)}\big)^3\Big)^{-1/2}\right] \\
& \times \Bigg\{\mathbb{I}_{\{k_0=0\}}\int_0^\infty\Bigg[\prod_{\substack{l\,:\,k_l=0\\ l\neq 0}} p_{n_l}(r)\Bigg]\Bigg[\prod_{l\,:\,k_l\neq 0} q_{i_1^{(l)}}(r)\Bigg]\exp\Bigg(-\frac{r^2}{2}\sum_{l\,:\,k_l\neq 0}\frac{1}{i_1^{(l)}}\Bigg)\mathrm{d}r \\
& \qquad + \sqrt{2\pi}\,\mathbb{I}_{\{k_0=1\}}\prod_{l\,:\,k_l=0}\frac{(2n_l-1)!!}{(2n_l)!!}\prod_{\substack{l\,:\,k_l\neq 0\\ l\neq 0}}\frac{1}{i_1^{(l)}}\Bigg\}.
\end{aligned}$$

Expected d-dimensional volume of the convex hull

- Recalling the Cauchy surface area formula: $\mathbb{E}|\mathcal{C}_{d-1}| = \frac{\kappa_{d-1}}{d\kappa_d}\mathbb{E}|\partial\,\mathcal{C}_d|$ leads to a formula for the expected (d-dimensional) volume of $\mathcal{C}_d$

Theorem

$$\mathbb{E}\,|\mathcal{C}_d| = d!\,\kappa_d\,(2\pi)^{-(d+1)/2} \sum_{\substack{k_0,\dots,k_m\ge 0\\ k_0\le n_0,\dots,k_m\le n_m\\ k_0+\dots+k_m=d+1}} \sum_{\substack{1\le i_1^{(l)}<\dots<i_{k_l}^{(l)}\le n_l\\ l=0,\dots,m\,:\,k_l>0}} \mathbb{E}|\mathrm{conv}\,\mathsf{T}_d|^2$$

$$\times \prod_{l\,:\,k_l\neq 0}\left[\frac{(2(n_l-i_{k_l}^{(l)})-1)!!}{(2(n_l-i_{k_l}^{(l)}))!!}\Big(i_1^{(l)}\big(i_2^{(l)}-i_1^{(l)}\big)^3\dots\big(i_{k_l}^{(l)}-i_{k_l-1}^{(l)}\big)^3\Big)^{-1/2}\right]$$

$$\times\Bigg\{\mathbb{I}_{\{k_0=0\}}\int_0^\infty\Big[\prod_{\substack{l\,:\,k_l=0\\ l\neq 0}} p_{n_l}(r)\Big]\Big[\prod_{l\,:\,k_l\neq 0} q_{i_1^{(l)}}(r)\Big]\exp\Big(-\frac{r^2}{2}\sum_{l\,:\,k_l\neq 0}\frac{1}{i_1^{(l)}}\Big)\mathrm{d}r$$

$$+\sqrt{2\pi}\,\mathbb{I}_{\{k_0=1\}}\prod_{l\,:\,k_l=0}\frac{(2n_l-1)!!}{(2n_l)!!}\prod_{\substack{l\,:\,k_l\neq 0\\ l\neq 0}}\frac{1}{i_1^{(l)}}\Bigg\}.$$

Affine Blaschke-Petkantschin formula

- Let $\mathbb{S}^{d-1}$ be the unit $(d-1)-$dimensional sphere, centered at the origin and equipped with the Lebesgue measure μ normalized to be probabilistic.
- For $u \in \mathbb{S}^{d-1}$, let $u^{\perp}$ be the linear hyperplane orthogonal to u.

Then, for any non-negative measurable function $h : (\mathbb{R}^d)^d \to \mathbb{R}$,

$$\int_{(\mathbb{R}^d)^d} h(x_1, \ldots, x_d) \mathrm{d}x_1 \ldots \mathrm{d}x_d$$

$$= d! \, \kappa_d \int_{\mathbb{S}^{d-1}} \int_0^\infty \int_{(\mathbf{u}^\perp + r\mathbf{u})^d} h(x_1, \ldots, x_d) \, |\mathrm{conv}\,(x_1, \ldots, x_d)| \times \lambda_{\mathbf{u}^\perp}(\mathrm{d}x_1) \ldots \lambda_{\mathbf{u}^\perp}(\mathrm{d}x_d) \, \mathrm{d}r \, \mathrm{d}\mu(\mathrm{d}u)$$

$$= d! \, \kappa_d \int_{\mathbb{S}^{d-1}} \int_0^\infty \int_{(\mathbf{u}^\perp)^d} h(x_1 + r\mathbf{u}, \ldots, x_d + r\mathbf{u}) \, |\mathrm{conv}\,(x_1, \ldots, x_d)| \times \lambda_{\mathbf{u}^\perp}(\mathrm{d}x_1) \ldots \lambda_{\mathbf{u}^\perp}(\mathrm{d}x_d) \, \mathrm{d}r \, \mathrm{d}\mu(\mathrm{d}u).$$

see eg Schneider & Weil (2008). *Stochastic and integral geometry*, Springer, Berlin.

Affine Blaschke-Petkantschin formula

We apply the B-P formula to compute:

$$\mathbb{E}[g(\mathsf{S}_d)\mathbb{I}_{\{\mathrm{conv}\,\mathsf{S}_d \in \mathcal{F}(\mathcal{C}_d)\}}] = \int_{(\mathbb{R}^d)^d} \mathbb{P}[\mathrm{conv}\,\mathsf{S}_d \in \mathcal{F}(\mathcal{C}_d) \,|\, \mathsf{S}_d = (\mathsf{x}_1, \dots, \mathsf{x}_d)] \times g(\mathsf{x}_1, \dots, \mathsf{x}_d) f_{\mathsf{S}_d}(\mathsf{x}_1, \dots, \mathsf{x}_d) \mathrm{d}\mathsf{x}_1 \dots \mathrm{d}\mathsf{x}_d,$$

where f_{S_d} is the joint density of S_d.

Affine Blaschke-Petkantschin formula

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$$\times\, g(\mathsf{x}_1, \ldots, \mathsf{x}_d) f_{\mathsf{S}_d}(\mathsf{x}_1, \ldots, \mathsf{x}_d)\mathrm{d}\mathsf{x}_1 \ldots \mathrm{d}\mathsf{x}_d,$$

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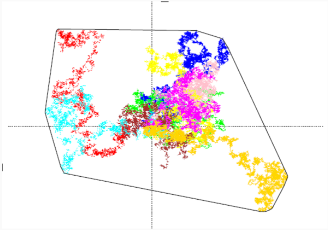

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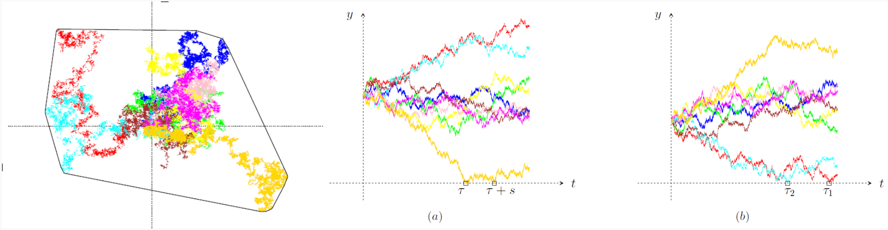
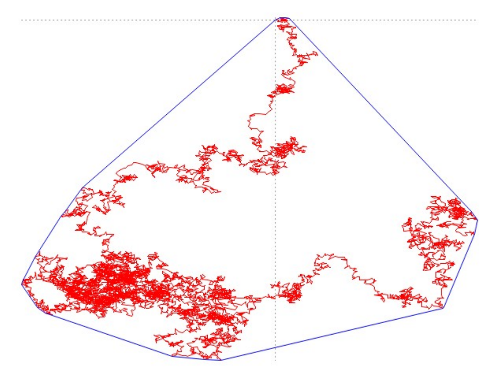


Brownian (and Lévy) convex hulls

Counting edges on the Brownian convex hull



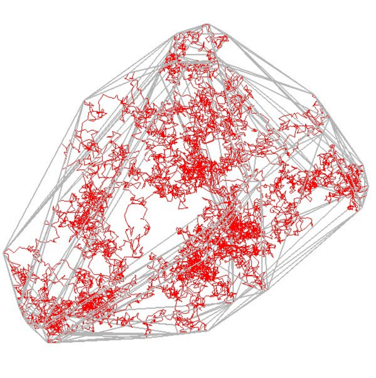
- in 2D a.s. $\partial \mathcal{C}$ minus its extreme points is a countable union of segments

El Bachir *L'enveloppe convexe du mouvement brownien* PhD thesis, (adv. G. Letac), UPS Toulouse (1983)

Evans *On the Hausdorff dimension of Brownian cone points* Math. Proc. Cambridge. Philos. Soc., 98, 343 (1985)

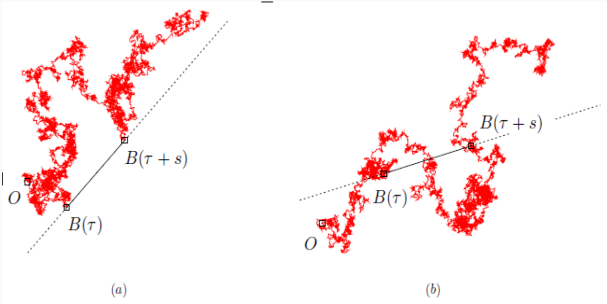
Cranston, Hsu, March *Smoothness of the convex hull of planar Brownian motion* Ann. Prob., 17, 1, 144 (1989)

Counting facets on the Brownian (and Lévy) convex hull



- in 2D a. s. $\partial\mathcal{C}$ minus its extreme points is a countable union of segments
- in dimension d: a.s. a countable union of (d-1)-dimensional facets

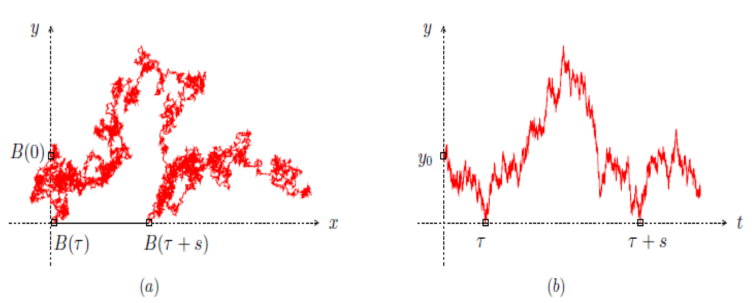
Brownian case: edge probability



Given times $s \in [\Delta t, T]$ and $\tau \in [0, T - s]$, path Γ_B may be found either:

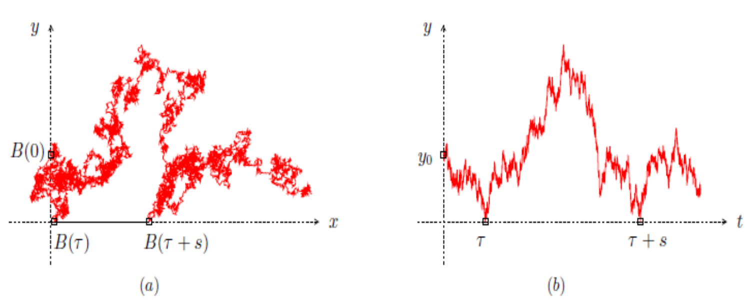
- to lie on one side only of the line through $B(\tau)$ and $B(\tau + s)$
- or, to cross this line.

In the s, τ-adapted frame...



... new y coordinate satisfies:

1. y starts at $y_0 > 0$ (say) and hits 0 for the first time at $t = \tau$;
2. y hits 0 again at $t = \tau + s$ but remains > 0 between $t = \tau$ and $t = \tau + s$;
3. y does not hit 0 after time $t = \tau + s$, it remains > 0 up to time $t = T$.



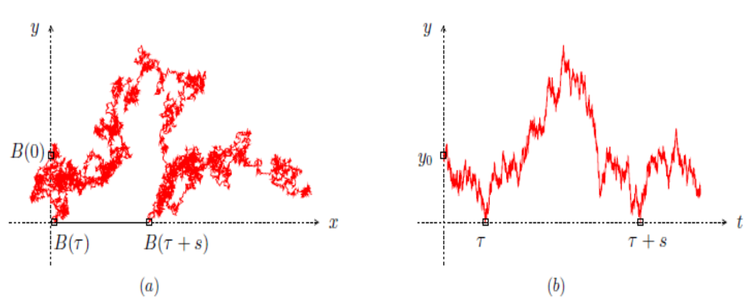
Split path, and obtain:

1. first part has a first-passage time equal to τ
   (or equivalently, in reverse time it is a Brownian meander of duration τ);

2. second part is a Brownian excursion of duration s;

3. third part is a Brownian meander of duration $T - (\tau + s)$.

Chung ***Excursions in Brownian motion*** **Ark. Mat., 14, 155 (1976)**

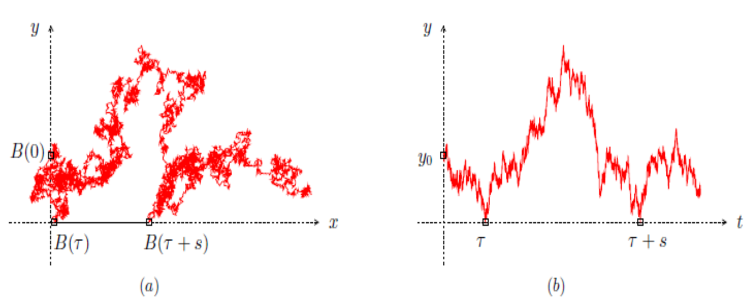


Let $\rho(s, \tau)$ be the "edge" probability density function (pdf), and introduce:

$$p(t; u, v) = \frac{1}{\sqrt{2\pi t}} \exp\left[-(v-u)^2/2t\right]$$

and

$$g(t; 0, v) = \frac{v}{\sqrt{2\pi t^3}} \exp\left[-v^2/2t\right], \text{ for } v > 0$$



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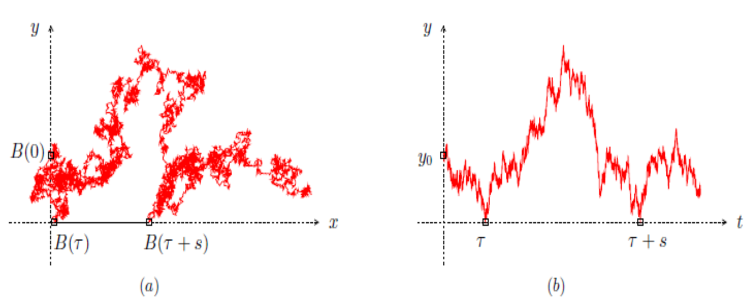
One should be able to express ρ in terms of p and g.

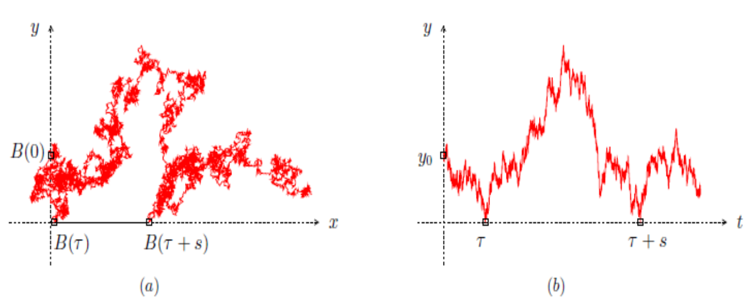
Chung *Excursions in Brownian motion* Ark. Mat., 14, 155 (1976)



Yes, indeed...:

$$\rho(s,\tau) = 2 \times \frac{\int_0^\infty g(\tau;0,y_0)\,\mathrm{d}y_0}{\int_0^\infty p(\tau;z_0,0)\,\mathrm{d}z_0} \times \frac{\int_0^\infty g(\kappa-\tau;0,r)\,g(\tau+s-\kappa;0,r)\,\mathrm{d}r}{\int_0^\infty p(\kappa-\tau;0,r')\,p(\tau+s-\kappa;r',0)\,\mathrm{d}r'}$$

$$\times \frac{\int_0^\infty g(T-(\tau+s);0,y_T)\,\mathrm{d}y_T}{\int_{-\infty}^\infty p(T-(\tau+s);0,z_T)\,\mathrm{d}z_T}$$



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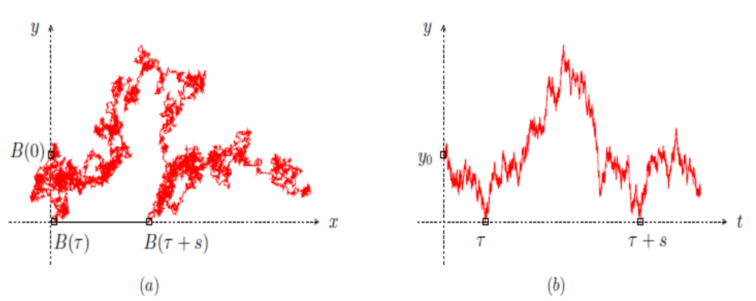
$$\times \frac{\int_0^\infty g(T-(\tau+s);0,y_T)\,\mathrm{d}y_T}{\int_{-\infty}^\infty p(T-(\tau+s);0,z_T)\,\mathrm{d}z_T}$$

↬ $$\rho(s,\tau) = \frac{2}{\pi s\sqrt{\tau[(T-s)-\tau]}}$$

A closer look at s-edges



$$\rho(s,\tau) = \frac{2}{\pi s\sqrt{\tau[(T-s)-\tau]}} = 2 \times \frac{1}{\pi\sqrt{\tau[(T-s)-\tau]}} \times \frac{1}{s}$$

$$= p^{0}_{(\tau)} \underbrace{f(\tau,s)}_{\text{indep. of } \tau} p^{(\tau+s)}_{T} = \underbrace{p^{0}_{(\tau)}\, p^{(\tau+s)}_{T}} \times f(s)$$

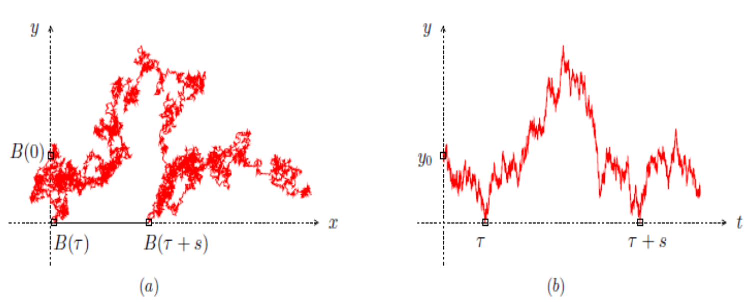
indep. of τ

pdf to hit $\mathbf{0}$ at τ

for a path of duration $T-s$

conditioned to have min. equal to $\mathbf{0}$

A closer look at s-edges

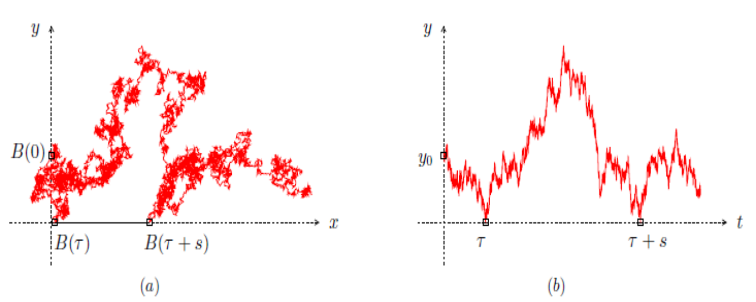


Upon integration over τ:

$$\int_{\tau=0}^{T-s} \rho(s,\tau) d\tau \, ds = 2 \overbrace{\left(\int_{\tau=0}^{T-s} p^{0}_{(\tau)} \; p^{(\tau+s)}_{T} \, d\tau \right)}^{1} f(s)\, ds$$

$$= 2 \; f(s) \; ds = 2 \times \frac{1}{s} \, ds$$

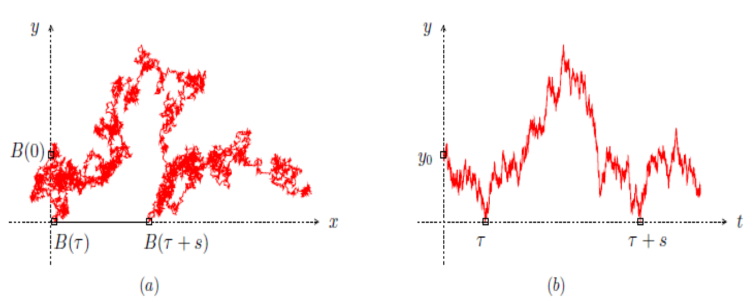
A closer look at s-edges



Sojourn time density...

$f(s) \rightarrow$ pdf of sojourn time in $\mathbb{R}_+^*$ for a process with duration s

A closer look at s-edges



Sojourn time density...

$f(s) \to$ pdf of sojourn time in $\mathbb{R}_+^*$ for a process with duration s

...is uniform

for many processes, this is **uniform** on $[0, s]$ and thus $f(s) = \frac{1}{s}$.

Knight *The uniform law for exchangeable and Lévy process bridges* Astérisque, 236, 171–188 (1996)
Baxter *A combinatorial lemma for complex numbers* Ann. Math. Stat., 32, 901 (1961)
Barndorff-Nielsen, Baxter *Combinatorial lemmas in higher dimensions* Trans. Amer. Math. Soc., 108 313 (1963)

Result for Brownian-Lévy motion in dimension 2

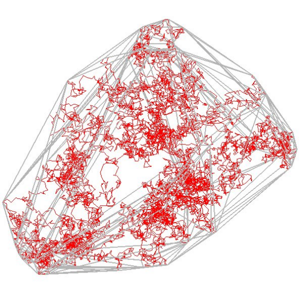
Average number of edges

$$
\begin{aligned}
\mathbb{E}\left[\mathcal{F}_T^{(2)}\right] &= \int_{\Delta t}^{T}\int_{0}^{T-s} \rho(s,\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&= 2\int_{\Delta t}^{T} \frac{\mathrm{d}s}{s} \\
&= 2\ln\left(\frac{T}{\Delta t}\right)
\end{aligned}
$$

R-F, **Physical Review E**, 89 **(5)**, **052112 (2014)**

R-F, **J. Phys. A: Math. Theor.**, 46, **015004 (2013)**

General d-dimensional case



In 3 dimensions

$$\mathbb{E}\left[\mathcal{F}_T^{(3)}\right] = 2\int_{\Delta t}^{T-\Delta t}\int_{\Delta t}^{T-k_1}\int_0^{T-(k_1+k_2)} dk_1\ dk_2\ d\tau\ \ldots$$

$$\ldots\ \underbrace{p_\tau^{(0)}\ p_T^{(\tau+k_1+k_2)}}_{\text{will integrate out to } 1}\ \underbrace{f(\tau,\tau+k_1)\ f(\tau+k_1,\tau+k_1+k_2)}_{=\frac{1}{k_1}\frac{1}{k_2},\ \text{indep. of } \tau}$$

General d-dimensional case

In 3 dimensions:

$$\mathbb{E}\left[\mathcal{F}_T^{(3)}\right] = 2 \iint\limits_{\substack{k_1+k_2\leq T \\ k_1,k_2\geq\Delta t}} \frac{dk_1\,dk_2}{k_1 k_2}$$

General d-dimensional case

In 3 dimensions:

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In d dimensions:

$$\mathbb{E}\left[\mathcal{F}_T^{(d)}\right] \;=\; 2 \int\cdots\int\limits_{\substack{k_1+\cdots+k_{d-1}\leq T\\ k_1,\ldots,k_{d-1}\geq \Delta t}} \frac{dk_1\cdots dk_{d-2}\,dk_{d-1}}{k_1\,k_2\cdots k_{d-1}}$$

Results for Brownian-Lévy motion in the d-dimensional case

In 3 dimensions:

$$\mathbb{E}\left[\mathcal{F}_T^{(3)}\right] = 2\left[\ln\left(T/\Delta t\right)\right]^2 + 4\left\{\ln\left(T/\Delta t\right)\ln\left(1-\Delta t/T\right) - \mathrm{Li}_2\left(1-\Delta t/T\right) + \pi^2/12\right\},$$

where Li_2 is the dilogarithm function.

In d dimensions:

$$\mathbb{E}\left[\mathcal{F}_T^{(d)}\right] \sim 2\left[\ln\left(\frac{T}{\Delta t}\right)\right]^{d-1}$$

This generalizes the 2-dimensional result and the discrete-time result.

NB one can also look at several (indep.) Brownian or Lévy walkers

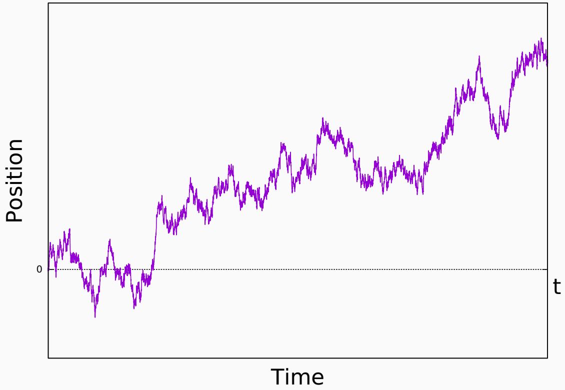
R-F and Wespi, Physical Review E, 95 (3), 032129 (2017)

Kabluchko, Vysotsky, Zaporozhets ***Convex hulls of RWs: Expected number of faces and face probabilities*** **Adv. in Math. 320, 595-629 (2017)**

R-F, J. Phys. A: Math. Theor., 46, 015004 (2013)

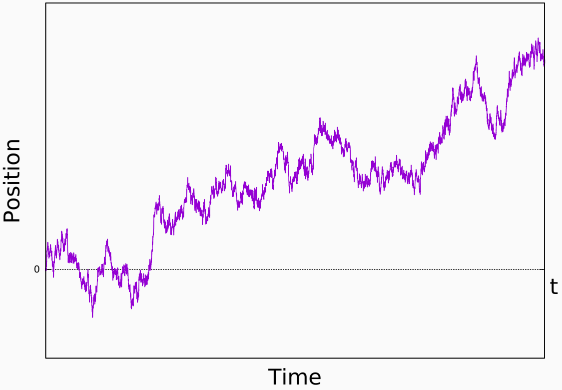
Some other extreme-value problems

A (toy) foraging problem (by P. Krapivsky)



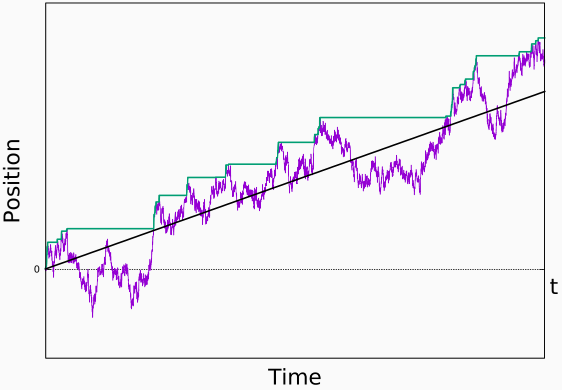
- “forager” on a line $\rightarrow$ Brownian walker with position $B(t)$
- one unit of “food” per unit length, no food replenishment
- “metabolism”: walker stockpiles, needs one unit of food per unit time
- survival probability?

One-sided version



- "forager" on a line $\rightarrow$ Brownian walker with position $B(t)$
- one unit of "food" per unit length, no food replenishment
- "metabolism": walker needs one unit of food per unit time
- food on > 0 side only

One-sided version: a hitting-time problem for the supremum



Letting

$$M(s) = \sup_{0 \le \tau \le s} B(\tau)$$

Survival probability is:

$$P(t) = \mathrm{Prob}\left(M(s) > s,\ \forall s \le t\right)$$

A hitting-time problem for the supremum

Survival probability $P(t) = \mathrm{Prob}\,(M(s) > s,\, \forall s \leq t)$

$$\downarrow$$

$$f(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\, P(t)$$

Probability density function (pdf) of extinction time

$$\|$$

Pdf of first hitting time for $M(s)$ on the diagonal, $\inf\limits_{s>0} \{M(s) - s = 0\}$

Idea: look at paths with $M(t) = t$

Path going extinct at $t \Rightarrow M(t) = t$, but...

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$$M(t) = t \not\Rightarrow \text{path going extinct at } t$$

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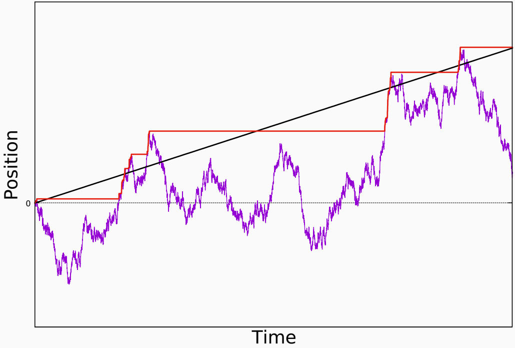
because path could have gone extinct before.



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$$f(t) = \overbrace{\mathrm{pdf}\,(M(t) = t)}^{\text{paths with } M(t)=t} \quad - \quad \underbrace{g(t)}_{\substack{\text{paths with } M(t)=t \\ \textit{and} \\ M(s)=s \text{ for some } s<t}}$$

Idea: look at paths with $M(t) = t$

Path going extinct at $t \Rightarrow M(t) = t$, but...

$$M(t) = t \not\Rightarrow \text{path going extinct at } t$$

because path could have gone extinct before.

$$f(t) = \overbrace{\sqrt{\frac{2}{\pi t}} \exp\left(-\frac{t}{2}\right)}^{\text{paths with } M(t)=t} - \underbrace{g(t)}_{\substack{\text{paths with } M(t)=t \\ \text{and} \\ M(s)=s \text{ for some } s<t}}$$

A path transformation (1)

Given a path with $M(t) = t$

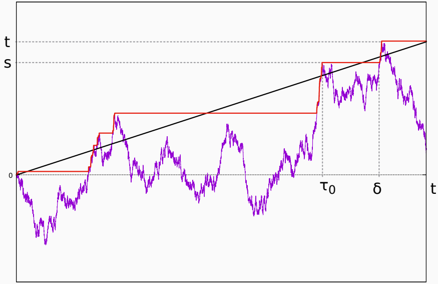
and $M(s) = s$ for some $s < t$, define:

- hitting time of $M(s) = s$:

  $\tau_0 = \inf \{r > 0, B(r) = M(s) \text{ i.e. } B(r) = s\}$

- first time level s is hit after τ_0:

  $\delta = \inf \{r > \tau_0, B(r) \geq M(s) \text{ i.e. } B(r) \geq s\}$



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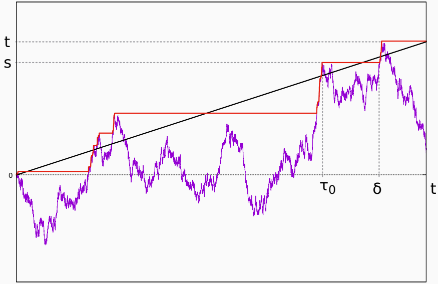

Then

- Note: **path between τ_0 and δ is a (downward) excursion**

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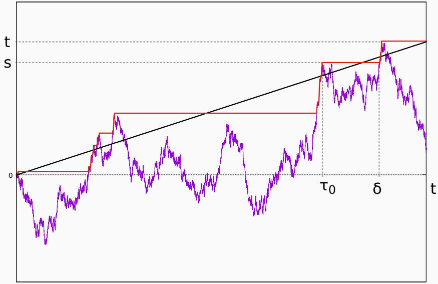
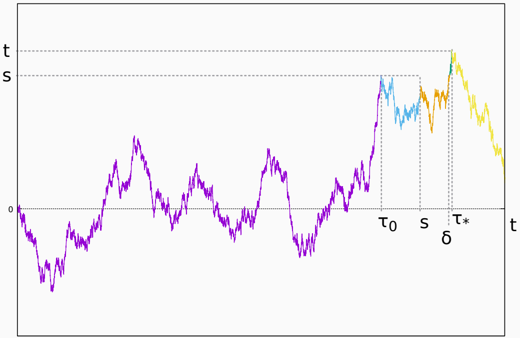


Then

- Note: **path between τ_0 and δ is a (downward) excursion**
- Idea : extract this excursion & use it to hit a new global maximum $> t$

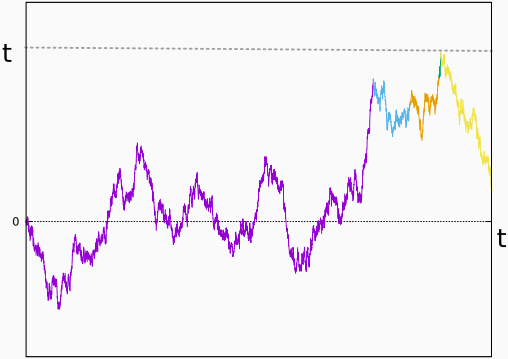
A path transformation (1)



- Define hitting time of the global maximum,

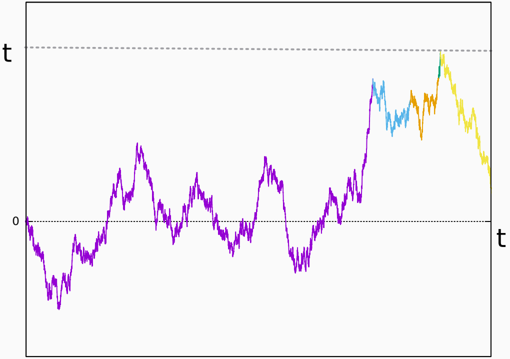
$$\tau_* = \inf\{r > 0, B(r) = M(t) \text{ i.e. } B(r) = t\}$$

A path transformation (1)



- extract excursion & bring "forward" (to τ_0) the $[\delta, \tau_*]$ part
- insert then the excursion transformed into an (upward) first passage bridge
- insert the final, post-τ_* part shifted upward as needed

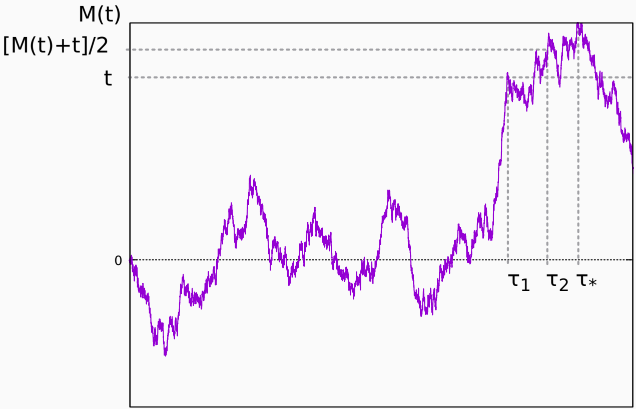
A path transformation (1)



- extract excursion & bring "forward" (to τ_0) the $[\delta, \tau_*]$ part
- insert then the excursion transformed into an (upward) first passage bridge
- insert the final, post-τ_* part shifted upward as needed

↬ obtain a path with global maximum $> t$

A path transformation (2)

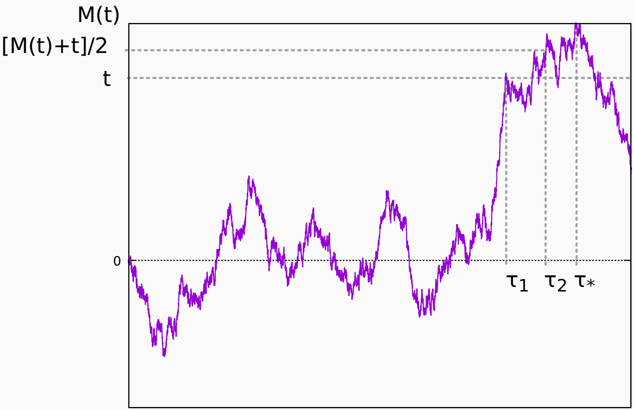


Start with a Brownian path having $M(t) > t$, and set:

$$\tau_1 = \inf\{s > 0, B(s) = t\}, \quad \tau_* = \inf\{s > 0, B(s) = M(t)\},$$

and $$\tau_2 = \inf\{s > 0, B(s) = [M(t) + t]/2\}.$$

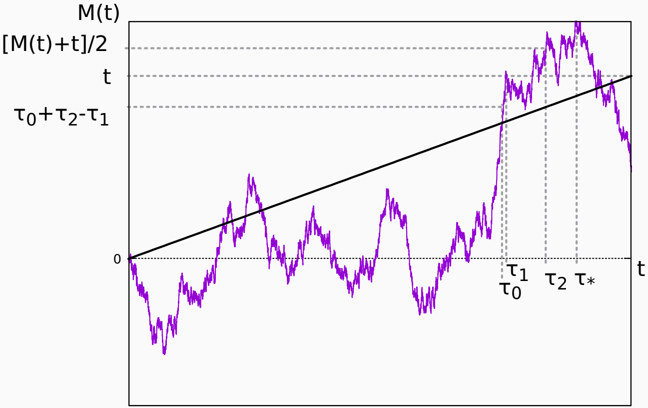
A path transformation (2)



Note that $B(0) - 0 = 0$ and $B(\tau_1) - \tau_1 = t - \tau_1 > \tau_2 - \tau_1$, so

$$\exists\, \tau_0 \in]0, \tau_1[\ \text{ s.t. } \tau_0 = \inf\{s > 0, B(s) - s = \tau_2 - \tau_1\}.$$

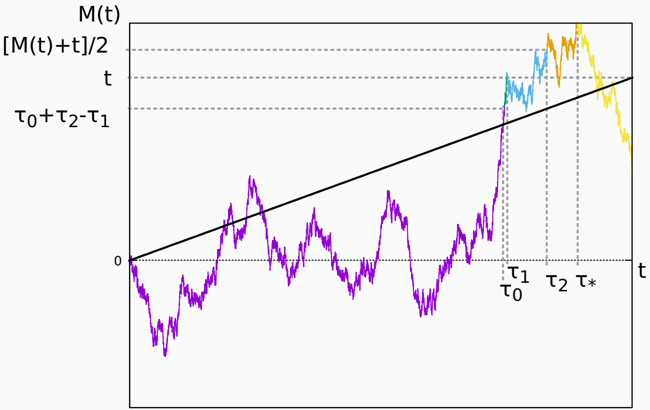
A path transformation (2)



Note that $B(0) - 0 = 0$ and $B(\tau_1) - \tau_1 = t - \tau_1 > \tau_2 - \tau_1$, so

$$\exists\, \tau_0 \in \,]0, \tau_1[\ \text{ s.t. } \tau_0 = \inf \{s > 0, B(s) - s = \tau_2 - \tau_1\}\,.$$

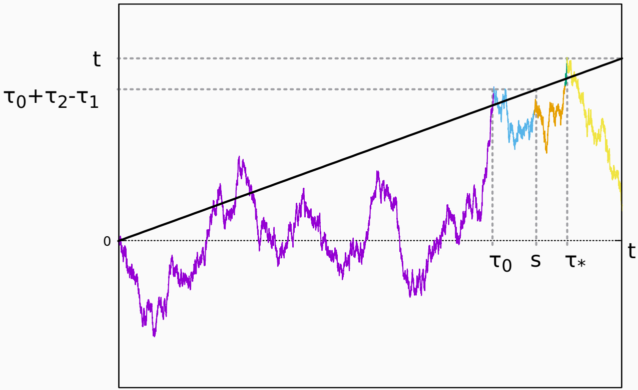
A path transformation (2)



Decompose the Brownian path as follows:

- take the τ_1 to τ_* part out,
- form an excursion of duration $\tau_* - \tau_1$ with subpath $[\tau_1, \tau_*]$
- insert excursion (downward) at time τ_0
- append then the $[\tau_0, \tau_1]$ part and the post-τ_* part

A path transformation (2)



↬ obtain a path with $M(t) = t$, "dying" (for sure) at time $s = \tau_0 + \tau_2 - \tau_1$.

PDF of extinction time in the one-sided case

Recall:

$$f(t) \;=\; \overbrace{\sqrt{\frac{2}{\pi t}}\exp\left(-\frac{t}{2}\right)}^{\text{paths with } M(t)=t} \;-\; \underbrace{g(t)}_{\substack{\text{paths with } M(t)=t \\ \textit{and} \\ M(s)=s \text{ for some } s<t}}$$

PDF of extinction time in the one-sided case

Now:

$$f(t) = \overbrace{\sqrt{\frac{2}{\pi t}}\exp\left(-\frac{t}{2}\right)}^{\text{paths with } M(t)=t} - \underbrace{g(t)}_{\text{paths with } M(t)>t}$$

PDF of extinction time in the one-sided case

That is,

$$f(t) \;=\; \sqrt{\frac{2}{\pi t}}\,\exp\left(-\frac{t}{2}\right) \quad - \quad \int_t^\infty \sqrt{\frac{2}{\pi t}}\,\exp\left(-\frac{m^2}{2t}\right)\,dm$$

PDF of extinction time in the one-sided case

Finally:

$$f(t) = \sqrt{\frac{2}{\pi t}} \exp\left(-\frac{t}{2}\right) - \operatorname{erfc}\left(\sqrt{\frac{t}{2}}\right)$$

PDF of extinction time in the one-sided case

Finally:

$$f(t) = \sqrt{\frac{2}{\pi t}} \exp\left(-\frac{t}{2}\right) - \operatorname{erfc}\left(\sqrt{\frac{t}{2}}\right)$$

Two other approaches:

- look at the reciprocal process of $M(s) - s$

  $\rightarrow$ this is a spectrally positive Lévy process

- show that the first passage time of $M(t) - t$
  is distributed like the sojourn time above 0 of the process $B(t) - t$

R.A. Doney *Hitting probabilities for spectrally positive Lévy processes*, Journal of the LMS, 2(3):566-576 (1991)

J-P Imhof *On the time spent above a level by Brownian motion with negative drift* Adv. in Appl. Prob., 18(4):1017-1018 (1986)

JRF, P. Salminen, P. Vallois *On a first hit distribution of the running maximum of Brownian motion* arXiv:2103.08358 (2021)

Conclusion & Perspectives

- "simple" problems with explicit solutions
- path decomposition & path transformation techniques
- higher order moments for convex hulls?
- first-passage time to affine barrier for Brownian range?...

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Many thanks for your attention!