Phantom distribution functions for maxima on random trees

EcoDep Conference 2021
17th September 2021

Adam Jakubowski
Nicolaus Copernicus University
Toruń, Poland

This is a work *in progress* joint with Paul Doukhan

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But first a bit of “self”-promotion

News from the Bernoulli Society

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- It is a matter of tradition that the BS Officers promote their Society.

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  - SpringerBriefs in Probability and Mathematical Statistics, Springer;
  - Semstat books, CRC/Taylor&Francis.
  - Latin American Journal of Probability and Mathematical Statistics;
  - SemStat Elements.

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    - Next: 11th World Congress, August 12-16, 2024, Bochum (Germany).

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- Many other sponsored and co-sponsored meetings, including successful Bernoulli - IMS Young Researchers Meetings.

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- 2020 Doeblin Prize (Nike Sun)
- 2020 BS New Researcher Award (Li-Cheng Tsai, Nina Holden, Xin Sun)
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- 2021 Ethel Newbold Award (Marloes Maathuis).
- 2021 BS New Researcher Award (Fang Han, Aaditya Ramdas, Anru Zhang).
- 2021 Bernoulli Presidential Invited Lecture (Markus Reiss).
- 2021 Bernoulli Journal Lecture (Johannes Schmidt-Hieber).

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- Willem van Zwet Medal for special service to the Bernoulli Society (Maria Eulália Vares).

Back to the topic: Classics

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- Let $X_1, X_2, \ldots$ be an iid sequence of random variables with marginal distribution function $F(x) = \mathbb{P}(X_1 \leqslant x)$. Let $M_n = \max_{1 \leqslant j \leqslant n} X_j$.

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- The traditional approach (Fisher & Tippet, Gnedenko, de Haan, ...):

$$\lim_{n\to\infty} \mathbb{P}((M_n - b_n)/a_n \leqslant x) = \lim_{n\to\infty} \mathbb{P}(M_n \leqslant a_n x + b_n) = H(x),\ x \in \mathbb{R}^1. \quad (*)$$

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- L. de Haan, A. Ferreira, **Extreme Value Theory. An Introduction**, Springer 2006. M.R. Leadbetter, G. Lindberg, H. Rootzén, **Extremes and related properties of random sequences and processes**, Springer 1986.

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- Every continuous distribution function is regular.

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$$G^n(v_n) \to \gamma, \quad H^n(v_n) \to \gamma.$$
- H is regular and its tail is equivalent to the tail of G, i.e.
$$G_* = H_* \quad \text{and} \quad \frac{1 - H(x)}{1 - G(x)} \to 1, \text{ gdy } x \to G_*-.$$

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- It follows that the phantom distribution function is determined uniquely *modulo* the equivalence of right tails.

Why to deal with PhDFs?

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- Let us recall the definition of a PhDF G for a stationary sequence $\{X_j\}$ with partial maxima $\{M_n\}$.

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Basic statement

If a stationary process $\{X_j\}$ admits a regular PhDF, then the asymptotics of maxima $\{M_n\}$ is completely determined by a single sequence of levels $\{v_n\}$ satisfying for some $\gamma \in (0, 1)$

$$\mathbb{P}(M_n \leqslant v_n) \to \gamma.$$

(For example, if $\gamma = 1/2$ then as v_n we can take the median of M_n).

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- One can find a number $\gamma \in (0,1)$ and a non-decreasing sequence $\{v_n\}$ such that
$$\mathbb{P}(M_n \leqslant v_n) \to \gamma,$$
and for each $T > 0$ the following condition $B_T(\{v_n\})$ holds:
$$\sup_{\substack{p,q\in\mathbb{N},\\ p+q\leqslant T\cdot n}} \left|\mathbb{P}(M_{p+q} \leqslant v_n) - \mathbb{P}(M_p \leqslant v_n)\mathbb{P}(M_q \leqslant v_n)\right| \to 0.$$

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$$\sup_{\substack{p,q\in\mathbb{N},\\ p+q\leqslant T\cdot n}} \left|\mathbb{P}(M_{p+q} \leqslant v_n) - \mathbb{P}(M_p \leqslant v_n)\mathbb{P}(M_q \leqslant v_n)\right| \to 0.$$

- One can find a number $\gamma \in (0,1)$ and a non-decreasing sequence $\{v_n\}$ such that on some dense subset $\mathbb{Q} \subset \mathbb{R}^+$

$$\mathbb{P}(M_{\lfloor nt \rfloor} \leqslant v_n) \to \gamma^t, \; t \in \mathbb{Q}.$$

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- If the covariance function r_n of a standard stationary Gaussian sequence satisfies $r_n \ln n \to \rho > 0$, then this sequence does not admit any PhDF.

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- For simplicity we define also

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  4. Gaussian fields satisfying the corresponding Berman condition.
- Wu & Samorodnitsky (SPA, 2020) give examples of calculation of the extremal index ($G = F^{\theta}$) for random fields with so-called “tail field".

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  - for $n = 1, 2, \ldots$ $\psi(n) \leqslant \psi(n+1)$ and $\psi(n) \neq \psi(n+1)$ (hence the sequence $\{\psi(n)^*\}$ is strictly increasing);
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Observation

Let G be a continuous DF. Then G is a PhDF for $\{X_{\mathbf{n}}\}$ if, and only if, G is a PhDF for $\{X_{\mathbf{n}}\}$ along every monotone curve.

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Theorem (J., Rodionov & Soja-Kukieła, Bernoulli, 2021)

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A stationary random field $\{X_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\}$ admits a continuous PhDF if, and only if, the following two conditions are satisfied.

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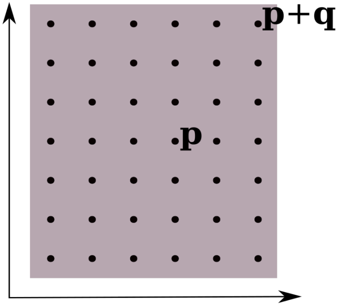
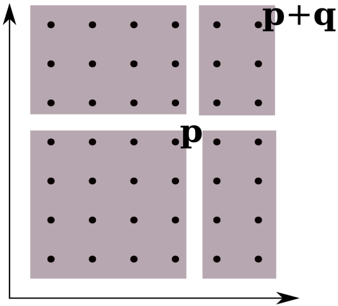
$$\mathbb{P}(M_{\mathbf{n}} \leqslant v_{\mathbf{n}}) \to \gamma, \text{ gdy } \mathbf{n} \to \infty.$$

- For every monotone curve ψ and every $T > 0$ the following condition $B_T^{\psi}(\{v_{\psi(n)}\})$ holds:

$$\beta_T^{\psi}(n) := \max_{\mathbf{p}(1)+\mathbf{p}(2)\leqslant T\psi(n)} \Big| \mathbb{P}\left(M_{\mathbf{p}(1)+\mathbf{p}(2)} \leqslant v_{\psi(n)}\right) - \prod_{\mathbf{i}\in\{1,2\}^d} \mathbb{P}\left(M_{(p_1(i_1),p_2(i_2),\ldots,p_d(i_d))} \leqslant v_{\psi(n)}\right) \Big| \xrightarrow[n\to\infty]{} 0.$$

Condition $B_T^{\psi}(\{v_{\psi(n)}\})$ and strong monotonicity





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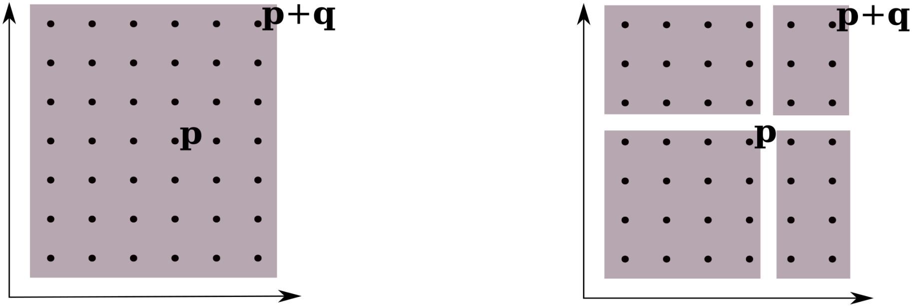

A comment

Suppose that F is continuous, choose $\gamma \in (0,1)$ and consider the corresponding quantiles:

$$v_{\mathbf{n}} = \inf\{x \,:\, \mathbb{P}\,(M_{\mathbf{n}} \leqslant x) = \gamma\}.$$

Then we have $\mathbb{P}(M_{\mathbf{n}} \leqslant v_{\mathbf{n}}) \to \gamma$ and the field of levels $\{v_{\mathbf{n}}\}$ is monotone, but there is no reason to expect it is strongly monotone.

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- Let $\mathbf{X} = \{X_{(i,j)}, (i,j) \in \mathbb{Z}^2\}$ be a Gaussian stationary random field, with zero expectations and unit variance and the covariance function

$$\mathbb{E}X_{(i,j)}X_{(0,0)} = r_{ij} = \eta_1(i)\eta_2(j)$$

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- Remark: $r_{\psi(n)} \cdot \ln n \to \gamma_1\gamma_2$.

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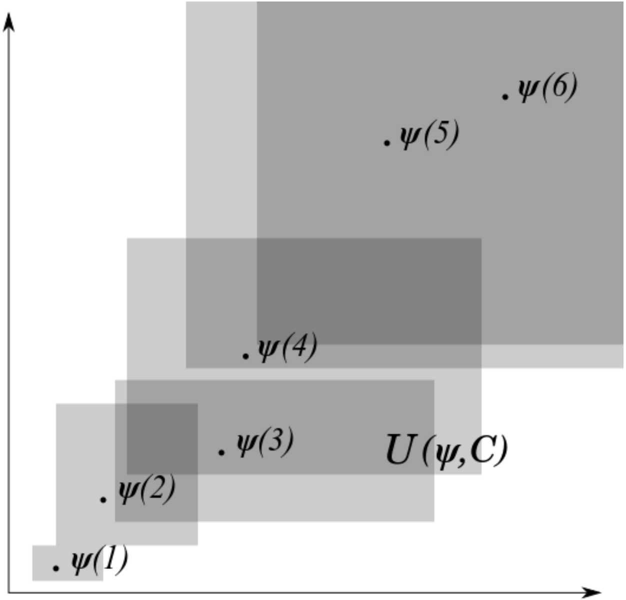
- For example, if $d = 2$ and $\{\phi(n)\}$ belongs to a cone (“sector") around the diagonal $\Delta(n) = (n, n)$, i.e. there is $C > 1$ such that for almost every $n \in \mathbb{N}$

$$C^{-1}n \leqslant \phi_1(n) \leqslant C\, n, \quad C^{-1}n \leqslant \phi_2(n) \leqslant C\, n,$$

then $\phi \in \mathcal{U}_\Delta$.

A graphical illustration for $U(\psi, C)$



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Definition

If $\{X_{\mathbf{n}}\}$ admits the same continuous PhDF along every curve $\phi \in \mathcal{U}_\psi$, then we say that $\{X_{\mathbf{n}}\}$ admits a ψ-directional PhDF.

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- Because it is (relatively) easy to obtain the condition

$$\mathbb{P}(M_{\psi(n)} \leqslant v_{\psi(n)}) \to \gamma, \text{ gdy } n \to \infty,$$

the sectorial (directional) PhDF is a useful tool in analysis of the asymptotics of maxima of random fields.

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- The convergence
$$\mathbb{P}(M_{(\lfloor ns \rfloor, \lfloor nt \rfloor)} \leqslant v_n) \to \gamma^{s \cdot t}, \ s, t \in \mathbb{R}^+,$$
does not imply the uniform convergence on $(\mathbb{R}^+)^2$.

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- Let $\mathbb{V}$ be a rooted tree. If $\mathbf{v} \in \mathbb{V}$, then $|\mathbf{v}|$ will denote the number of generation of $\mathbf{v}$ with respect to the root $\mathbf{r}$ (the length of the unique path connecting $\mathbf{v}$ and $\mathbf{r}$).

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- Let $\{X_{\mathbf{v}}\}_{\mathbf{v}\in\mathbb{V}}$ be a stochastic process indexed by $\mathbb{V}$. Define the partial maxima over branches:

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- Let G be a continuous distribution function. It is natural (motivated by the iid case) to say that G is a phantom distribution function for $\{X_{\mathbf{v}}\}_{\mathbf{v}\in\mathbb{V}}$, if

$$\sup_{x\in\mathbb{R}^1} \left|\mathbb{P}(M_{\mathbf{v}} \leqslant x) - G(x)^{|\mathbf{v}|}\right| \to 0,$$

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  when $|\mathbf{v}| \to +\infty$.
- Notice that $|\mathbf{v}| \to \infty$ implies that $\mathbb{V}$ is infinite.

Phantom distribution functions on trees

Phantom distribution functions on trees

Theorem

Suppose that $\mathbb{V}$ is an infinite rooted tree.
Then $\{X_{\mathbf{v}}\}_{\mathbf{v}\in\mathbb{V}}$ admits a continuous distribution function G if, and only if, there exist a number $\gamma \in (0,1)$ and a non-decreasing sequence of levels $\{u_n\}$ such that

$$\mathbb{P}(M_{\mathbf{v}} \leqslant u_n) - \gamma^{|\mathbf{v}|/n} \to 0,$$

uniformly in $\mathbf{v} \in \mathbb{V}$.

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- Let $\mathbb{B} = \{\mathbf{r}, \mathbf{v}_1, \mathbf{v}_2, \ldots\} \subset \mathbb{V}$ be a branch. It is natural to call G a phantom distribution function for $\{X_{\mathbf{v}}\}_{\mathbf{v}\in\mathbb{V}}$ along $\mathbb{B}$, if

$$\sup_{x\in\mathbb{R}^1} \left|\mathbb{P}(M_{\mathbf{v}_n} \leqslant x) - G(x)^n\right| \to 0,$$

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- Is it a (global) PhDF for $\{X_\mathbf{v}\}_{\mathbf{v}\in\mathbb{V}}$? In general not.
- The additional property we need is a kind of compactness of branches.

  For every sequence $|\mathbf{v}_n| \to \infty$ there exists a branch $\mathbb{B}$ containing an infinite number of elements of $\{\mathbf{v}_n\}$

PhDF on trees and PhDF along branches

PhDF on trees and PhDF along branches

Theorem

Suppose that all branches of a rooted tree $\mathbb{V}$ are infinite and the compactness property holds.
Then $\{X_{\mathbf{v}}\}_{\mathbf{v}\in\mathbb{V}}$ admits a continuous distribution function G if, and only if, there exist a number $\gamma \in (0,1)$ and a non-decreasing sequence of levels $\{u_n\}$ such that along every branch $\mathbb{B}$

$$\sup_{\mathbf{v}\in\mathbb{B}} \left|\mathbb{P}(M_{\mathbf{v}} \leqslant u_n) - \gamma^{|\mathbf{v}|/n}\right| \to 0.$$