Scaling features of two special Markov chains with total disasters

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Master2 Lecture, November 23, 2020

Introduction

MC: Given current pop. size is $x \in \mathbb{N}_0$, some pop. either grows by 1 unit (w.p. p_x) or collapses w.p. $q_x = 1 - p_x$. Collapse prob. q_x may be a ↓ or an ↑ function of x.

- In the latter ↑ case: large pops. are more susceptible and vulnerable to the black sheep and so quite unlikely to grow large. The case for a model with $q_x \sim 1 - \alpha x^{-\beta}$, $x \to \infty$. Think of building a house of cards: adding a new card to a house of cards of size x is more likely to lead to a collapse of the whole structure if x is already large. In such situation, process is always positive recurrent with light-tailed invariant measure (stretched exponential).

- In former ↓ opposite case (obtained while switching the roles of p_x and q_x, Wall duality), large pops. are getting more and more immune to black sheep and, after having successfully passed intermediate steps or not, large pop. sizes will stabilize in the end.

Introduction, c'td

2 special cases of catastrophe MC in latter class with total disasters, both DT and CT.

Depending on parameters range, the 2 models show up a recurrence/transience transition and, in critical case, a positive/null recurrence transition. Collapse transition probs. chosen so that models are exactly solvable and, in case of positive recurrence, intimately related to the extended Sibuya and Pareto-Zipf distributions whose divisibility and self-decomposability properties are shown relevant.

Study includes: existence and shape of the invariant measure, time-reversal, return time to origin, contact prob. at the origin, extinction prob., height and length of the excursions, a renewal approach to the fraction of time spent in the catastrophic state, scale function, first time to collapse and first-passage times, divisibility properties.

Relation to: Machine replacement in Queuing Systems. Physics of SOC (sandpile growth processes): long accumulation of grains brutally interrupted by critical avalanche. Stress-release explains power-law distributions of earthquakes sizes. Models for pop. growth subject to rare catastrophic extinction events. Crashes in finance, where pomp periods alternate periods of scarcity (Joseph and Noah effects).

General setup of catastrophe models

For any fixed x integer and U a $[0,1]$-valued rv, first consider the rv

$$U \circ x = \sum_{i=1}^{x} B_i(U),$$

where $(B_i(U), i \geq 1)$ is an iid sequence of Bernoulli rvs with $\mathbb{P}(B_i(U) = 1) = U$, random ($U \circ x$ is Bernoulli thinning of x). Clearly

$$\begin{aligned} \mathbb{E}\left(z^{U \circ x}\right) &= \mathbb{E}\left[(1 - U(1-z))^{x}\right], \text{ equivalently} \\ \mathbb{P}(U \circ x = y) &= \binom{x}{y} \mathbb{E}\left[U^{y}(1-U)^{x-y}\right], \; 0 \leq y \leq x \end{aligned}$$

and support of $U \circ x$ is $\{0, ..., x\}$.

$$\begin{aligned} &\mathbb{P}(U \circ x = y) = \tbinom{x}{y} u^{y}(1-u)^{x-y} \text{ if } U \overset{d}{\sim} \delta_u, \; u \in (0,1) \\ &\mathbb{P}(U \circ x = y) = \delta_{y,0} \text{ if } U \overset{d}{\sim} \delta_0 \\ &\mathbb{P}(U \circ x = y) = \delta_{y,x} \text{ if } U \overset{d}{\sim} \delta_1 \\ &\mathbb{P}(U \circ x = y) = \tfrac{1}{x+1} \text{ if } U \text{ is uniform.} \\ &\mathbb{P}(U \circ x = y) = \tbinom{x}{y} \tfrac{B(\alpha+y, \beta+x-y)}{B(\alpha,\beta)} \text{ if } U \overset{d}{\sim} \text{Beta}(\alpha,\beta), \; \alpha, \beta > 0. \end{aligned} \qquad (1)$$

General setup 'ctd

Let $(\xi_n;\ n \geq 1)$ be a sequence of iid rvs taking values in $\mathbb{N} = \{1, 2, 3, \ldots\}$. Discrete time-homogeneous Markov chain (MC) $X := (X_n; n \geq 0)$ with state-space $\mathbb{N}_0$ and non-homogeneous spatial transition probabilities characterized by:

• given $X_n = x \in \{1, 2, \ldots\}$, the increment of X_n is

$$\begin{array}{l} \xi_{n+1} \text{ with prob. } p_x \\ U \circ (x-1) - x \text{ with prob. } q_x. \end{array} \tag{2}$$

• given $X_n = 0$, increment of X_n is $+1$ with prob. $p_0 \leq 1$ and 0 with prob. $q_0 = 1 - p_0$.

If $\xi \overset{d}{\sim} \delta_1$, X is a skip-free to the right MC.

The stochastic transition matrix of model (2) is $P = [P(x,y)]$ where

$$\begin{aligned} P(x,y) &= q_x \mathbb{P}(U \circ (x-1) = y) \text{ if } 0 \leq y < x \\ P(x,y) &= p_x \mathbb{P}(\xi = y - x) \text{ if } y > x \end{aligned}$$

If both $\xi \overset{d}{\sim} \delta_1$ and $U \overset{d}{\sim} \delta_1$, P is the transition matrix of a general BD MC with tridiagonal Jacobi transition matrix.

Model with total disaster is one for which $U \overset{d}{\sim} \delta_0$ and $\xi \overset{d}{\sim} \delta_1$ (skip-free to the right).

Catastrophe: continuous-time

A continuous-time version $\overline{X}$ of the latter model is defined by the transition rate matrix $Q = D_{\mathbf{r}}(P - I)$ for some rate vector $\mathbf{r} = (r_0, r_1, \ldots, r_x, \ldots)$. . Given the chain $\overline{X}$ is in state x, a move occurs at exponential rate r_x. With probability p_x, the chain $\overline{X}$ moves up and it will be to a position $y > x$ with probability $p_x \mathbb{P}(\xi = y - x)$. It moves down with probability q_x and it will be to a position $0 \leq y < x$ with probability $q_x \mathbb{P}(U \circ (x-1) = y)$. Given a move occurs from state x, it is an up-move at rate $r_x^+ = r_x p_x$ and a down-move at rate $r_x^- = r_x q_x$ and

$$r_x^+ / r_x \text{ is the probability that a move-up occurs first from state } x$$
$$r_x^- / r_x \text{ is the probability that a move-down occurs first.}$$

Variants of such models were considered in Neuts, Anderson, (discrete-time with p_x, q_x independent of x) and Brockwell, Anderson (continuous-time, with $r_x^+ = a + bx$, $r_x^- = dx$ affine functions of x with $a, b \geq 0$ and $d > 0$, defining respectively the immigration rate and birth and death rates per capita).

Total disaster: Model 1

Let $\beta > 0$, $\nu > -1$ and $0 < \alpha < \nu + 1$. Consider the discrete time-homogeneous Markov chain (MC) $X := (X_n; n \geq 0)$ with state-space $\mathbb{N}_0 = \{0, 1, ...\}$ and non-homogeneous spatial transition probabilities characterized by:

• given $X_n = x \in \{1, 2, ...\}$, the increment of X_n is

$$\begin{aligned} &+1 \text{ with probability}: \ p_x = 1 - \alpha / \left(\nu + x^\beta\right) \\ &-x \text{ with probability}: \ q_x = \alpha / \left(\nu + x^\beta\right). \end{aligned} \tag{3}$$

• given $X_n = 0$, the increment of X_n is $+1$ with probability $p_0 \leq 1$ and 0 with probability $q_0 = 1 - p_0$.

Associated stochastic transition matrix: $P = [P(x, y)]$, $(x, y) \in \mathbb{N}_0^2$ with $P(x, 0) = q_x$ and $P(x, x+1) = p_x$, $x \geq 0$.

X is stochastically monotone.

If $\beta = 1$, X is ergodic but not time-reversible (no detailed balance).

Total disaster MC: continuous-time

Let $\lambda \in (-\infty, +\infty)$ and transition rate matrix Q of a CT MC process $\overline{X}(t)$:

$$Q = D_{\mathbf{r}}(P - I) \tag{4}$$

with $D_{\mathbf{r}}$ =diag($\mathbf{r}$) diagonal matrix formed from the rate vector $\mathbf{r} = (r_0, r_1, ..., r_x, ...)$ with $r_x = r_0(x+1)^{\lambda}$, $x \geq 0$, $r_0 > 0$. Clearly, X_n is the embedded MC of $\overline{X}(t)$.

Growth rate of transition $x \to x+1$ is: $r_x p_x$ with $r_x p_x \sim r_0 x^{\lambda}$ for large x, while the one of collapse transition $x \to 0$ is $r_x q_x$ with $r_x q_x \sim \alpha r_0 x^{\lambda-\beta}$ for large x. Swift chain: If $\lambda > 0$, $\alpha x^{\lambda-\beta} \ll x^{\lambda}$ always and if $0 < \lambda < \beta$, $\overline{X}$ moves up by one unit frequently, while its collapse becomes increasingly rare. If $\lambda < 0$ (lazy chain), growth rate $r_x p_x \sim r_0 x^{\lambda}$ is small for large x while collapse rate $r_x q_x \sim \alpha r_0 x^{\lambda-\beta}$ is still smaller. In all cases: collapse rates become small compared to the growth ones.
Let $P(t)$ standard Poisson process with intensity $t \geq 0$. Let $Z(t) = X_{P(t)}$ be the chain X_n subordinated to $P(t)$.
Then $Z(t)$ is a CT Markov chain with transition rate matrix $P - I$. And

$$\overline{X}(t) = Z\left(\int_0^t r_{\overline{X}_s} ds\right)$$

Total disaster MC: continuous-time

It has infinitesimal backward generator $G_{\overline{X}}$ whose action on real-valued bounded functions h on $\mathbb{N}_0 = \{0, 1, 2, ...\}$ is

$$G_{\overline{X}}h(x) = r_x \{((h(x+1) - h(x)) p_x + (h(0) - h(x)) q_x)\},$$

$$\mathbb{E}_x h\left(\overline{X}(t)\right) = \mathbb{E}_x h\left(\overline{X}(0)\right) + \int_0^t \mathbb{E}_x \left(G_{\overline{X}}h\right)\left(\overline{X}(s)\right) ds.$$

The range $\lambda > 0$ ($\lambda < 0$) accounts for the fact that moves of $\overline{X}$ get frequent (respectively rare) when the height x of $\overline{X}$ gets large, and given a move has occurred and x is large, $\overline{X}$ grows by one unit with large probability $\sim 1 - \alpha x^{-\beta}$ or undergoes a catastrophic event with small complementary probability $\sim \alpha x^{-\beta}$. Such transition mechanisms favor large values of $\overline{X}$. This chain is irreducible and aperiodic, either transient or recurrent (possibly then either positive or null recurrent).

Catastrophe: continuous-time

(i) When transient ($\beta > 1$), the process $\overline{X}$ is either explosive ($\lambda > 1$) or non-explosive ($\lambda \leq 1$). When $\beta > 1$, after a finite number of returns to 0, $\overline{X}$ drifts to ∞. And (using arguments on Yule process) it explodes if and only if $\lambda > 1$. The process $\overline{X}$ has no non-trivial ($\neq \mathbf{0}$) invariant measure.

(ii) When it is recurrent ($\beta \leq 1$), the process $\overline{X}$ is:

- recurrent positive if $\beta < 1$, with invariant measure showing a stretched exponential behavior.

- recurrent positive if $\beta = 1$ and $\alpha + \lambda > 1$, recurrent null if $\beta = 1$ and $\alpha + \lambda \leq 1$. The invariant measure of $\overline{X}$ is of power-law type with index $\alpha + \lambda$. The critical value $\beta = 1$ separates a recurrent phase ($\beta < 1$) from a transient phase ($\beta > 1$).

DT total disaster: invariant measure

$\pi' = \pi' P$. Non-trivial row-vector $\pi' \equiv (\pi_0, \pi_1, ..)$ exists iff $C_1 \equiv \sum_{y=1}^{\infty} q_y = \infty$.
-If in addition, $C_2 \equiv \sum_{x \geq 1} \prod_{y=0}^{x-1} p_y < \infty$, the invariant measure is unique and is a proper invariant probability measure.

$$\pi_x = \frac{\prod_{y=0}^{x-1} p_y}{1 + C_2}, \; x \geq 0.$$

- When $C_2 = \infty$, the measure solution exists but it is not a prob. as its total mass $\pi_0 (1 + C_2)$ sums to infty. SO HERE:
- $C_1 < \infty$ iff $\beta > 1$: MC is transient with no non-zero invariant measure.
- If $\beta < 1$, then $C_1 = \infty$ and $C_2 < \infty$: the MC is positive recurrent. For large x:

$$\pi_x \sim \prod_{y=0}^{x-1} \left(1 - \frac{\alpha}{\nu + y^{\beta}}\right) \sim e^{-\alpha \sum^x y^{-\beta}} \sim e^{-\frac{\alpha}{1-\beta} x^{1-\beta}}$$

with stretched exponential behaviour.
- In the critical case $\beta = 1$:

$$\pi_x \sim \prod_{y=0}^{x-1} \left(1 - \frac{\alpha}{\nu + y}\right) \sim e^{-\alpha \sum^x y^{-1}} \sim x^{-\alpha}$$

with power-law(α) behaviour. The DT chain is positive recurrent if $\alpha > 1$, null recurrent if $\alpha \leq 1$.

Return time to origin

$$\mathbb{P}\left(\tau_{0,0}=x+1\right)=q_x\prod_{y=0}^{x-1}p_y. \tag{5}$$

If $C_1 \equiv \sum_{y=1}^{\infty} q_y < \infty$, X_n is transient with $\mathbb{P}\left(\tau_{0,0}<\infty\right)<1$. If $C_1=\infty$, X_n is recurrent. If $C_1=\infty$ and $C_2 \equiv \sum_{x\geq 1}\prod_{y=0}^{x-1}p_y < \infty$, X_n is positive recurrent with $\mu := \mathbb{E}\left(\tau_{0,0}\right)=1/\pi_0=1+C_2<\infty$. If $C_1=C_2=\infty$, X_n is null recurrent with $\tau_{0,0}<\infty$ almost surely (a.s.) and $\mathbb{E}\left(\tau_{0,0}\right)=\infty$. In positive recurrent case,

$$\pi_x=\frac{\prod_{y=0}^{x-1}p_y}{1+C_2}=\frac{\mathbb{P}\left(\tau_{0,0}>x\right)}{\mu},\ x\geq 0.$$

The pgf of $\tau_{0,0}$ reads ($\tau_{0,0}-1$ is also the height H of an excursion)

$$\begin{aligned}\phi_{0,0}(z) &= \sum_{x\geq 0} q_x z^{x+1}\prod_{y=0}^{x-1}p_y = z\left(q_0+p_0\psi_0(z)\right)\\ &= z\left(q_0+p_0\frac{\alpha z}{\nu+1}\cdot F\left(1,\nu+1-\alpha;\nu+2;z\right)\right).\end{aligned}$$

$\psi_0(z)$ is pgf of an extended Sibuya rv $S\geq 1$ with parameters (α,ν). The shifted rv $S-1$ (with pgf $z^{-1}\psi_0(z)$) is discrete-SD (F Gauss HG function).

Positive recurrence, divisibility of invariant measure?

$$\frac{\pi_x}{1-\pi_0} = \frac{\pi_0 p_0}{1-\pi_0}\prod_{y=1}^{x-1}\left(1-\frac{\alpha}{\nu+y}\right) = \frac{\alpha-1}{\nu}\prod_{y=1}^{x-1}\left(1-\frac{\alpha}{\nu+y}\right)$$
$$= \frac{\alpha-1}{\nu-1+x}\prod_{y=1}^{x-1}\left(1-\frac{\alpha-1}{\nu-1+y}\right),\ x\geq 1$$

showing that $Y_\infty := X_\infty \mid X_\infty \geq 1 \overset{d}{\sim} \text{Sibuya}(\alpha-1,\nu-1)$ so with

$$\mathbb{E}\left(z^{Y_\infty}\right) = \frac{(\alpha-1)\,z}{\nu}\cdot F\left(1,\nu+1-\alpha;\nu+1;z\right) =: \psi_\infty\left(z\right)$$
$$\mathbb{E}\left(z^{X_\infty}\right) = \pi_0+\left(1-\pi_0\right)\psi_\infty\left(z\right).$$

X_∞ is a Bernoulli mixture of the two rvs $Y_\infty := X_\infty \mid X_\infty \geq 1$ and (say) Y_0 which is degenerate at 0 so with $Y_0 \overset{d}{\sim} \delta_0$. As extended Sibuya$(\alpha-1,\nu-1)$, shifted Sibuya rv

$$Y_\infty - 1 := \left(X_\infty - 1 \mid X_\infty \geq 1\right) \text{ is discrete SD.}$$

Is X_∞ itself ID (compound-Poisson)? discrete-SD? It turns out that X_∞ is ID and possibly SD if p_0 is small enough (else π_0 large enough).

Time-reversal

Assume X_n is recurrent. The catastrophe MC is not time-reversible as detailed balance does not hold. Let $\overleftarrow{P} \neq P$ be the transition matrix of the process $X_n^{\leftarrow}$ which is X_n backward in time. With $'$ denoting matrix transposition and $D_\pi = \text{diag}(\pi_0, \pi_1, \ldots)$, we have

$$\overleftarrow{P} = D_\pi^{-1} P' D_\pi .$$

The only non-null entries of $\overleftarrow{P}$ are its first row with $\overleftarrow{P}_{0,0} = q_0$ and $\overleftarrow{P}_{0,x} = q_x \prod_{y=0}^{x-1} p_y$ if $x \geq 1$ and the lower diagonal whose entries are all ones. Starting from $X_0^{\leftarrow} = x$, the process $X_n^{\leftarrow}$ decays linearly till it first hits 0 and once in state 0, $X_n^{\leftarrow}$ ends up jumping abruptly upward (after some latency time if $q_0 > 0$). Equivalently, $X_n^{\leftarrow}$ undergoes a jump of amplitude $x \geq 0$ with probability $q_x \prod_{y=0}^{x-1} p_y$ before diminishing again and again to 0. We have;

$$\mathbb{P}\left(\tau_{0,0} - 1 = x\right) = q_x \prod_{y=0}^{x-1} p_y$$

so the jump's amplitude is the one of $\tau_{0,0} - 1$ which is also the height H of an excursion.

Balance between shrinking against immigration.

Fraction of time spent in catastrophic state

A 'trivial' length 1 excursion appears whenever for some n, $X_n = X_{n+1} = 0$ (an event with prob. q_0). If $X_n = 0$ and $X_{n+1} \neq 0$, a 'true' excursion with length (say $\tau_{0,0}^{+}$) at least 2 starts from n, ending up when X first revisits 0. Time n for which $X_{n-1} \neq 0$ and $X_n = 0$. Distribution of the time elapsed from n, say $\Delta = N - n$ for which $X_{N-1} \neq 0$ and $X_N = 0$ for the first time $N > n$ again?

$$\Delta = \tau_{0,0}^{0} + \tau_{0,0}^{+},$$

where $\tau_{0,0}^{0} \overset{d}{\sim} \text{Geo}(p_0)$ is idle period (number of consecutive trivial excursions) and $\tau_{0,0}^{+} \geq 2$ length of a true excursion with: $\mathbb{P}\left(\tau_{0,0}^{+} = x+1\right) = q_x \prod_{y=1}^{x-1} p_y$, $x \geq 1$ (busy period). Lengths $\tau_{0,0}^{0}$, $\tau_{0,0}^{+}$ are mutually indep.

$$\tau_{0,0}^{+} - 1 \overset{d}{=} H^{+} = \inf\left(x \geq 1 : B_x\left(\alpha, \nu\right) = 1\right), \tag{6}$$

where $\left(B_x\left(\alpha, \nu\right);\ x \in \mathbb{N}_0\right)$ is a sequence of independent Bernoulli rvs obeying $\mathbb{P}\left(B_x\left(\alpha, \nu\right) = 1\right) = \alpha / \left(\nu + x\right) = q_x$.

APPLICATION (+ recurrent case, $\beta = 1$, $\alpha > 1$): $C_2^{+} = \sum_{x \geq 1} \prod_{y=1}^{x-1} p_y = C_2 / p_0 < \infty$

$$\mathbb{E}\left(\tau_{0,0}^{0}\right) = p_0 / q_0 \text{ and } \mathbb{E}\left(\tau_{0,0}^{+}\right) =: \mu^{+} = 1 + C_2^{+} = 1 + \frac{\nu}{\alpha - 1} > 2.$$

gives the average contribution to Δ of the 2 components.

Probability of extinction

Force state 0 to be absorbing and discard this state: P to $\overline{P}$. Let ϕ_x, $x \geq 1$ be the prob. that state 0 is hit in finite time given the chain started at x. The ϕ_xs obeys the recurrence: $\phi_x = q_x + p_x \phi_{x+1}$, else $1 - \phi_{x+1} = \frac{1}{p_x}\left(1 - \phi_x\right)$.

$$1 - \phi_x = \frac{1}{\prod_{y=1}^{x-1} p_y} \left(1 - \phi_1\right).$$

The formal solution is also $\phi = \left(I - \overline{P}\right)^{-1} \mathbf{q}$, involving the resolvent of $\overline{P}$ which is computable:

$$\phi_x = \sum_{y \geq x} q_y \prod_{y'=x}^{y-1} p_{y'}.$$

(i) If $C_1 < \infty$, the MC is transient and, with

$$\tau_{x,0} = \inf\left(n \geq 1 : X_n = 0 \mid X_0 = x\right), \; x \geq 0,$$

$\mathbb{P}\left(\tau_{x,0} = \infty\right) = \prod_{y \geq x} p_y > 0$. The chain X started at $x \geq 0$ has probability $\phi_x = 1 - \prod_{y \geq x} p_y < 1$ to undergo a first extinction.
(ii) If $C_1 = \infty$, the MC is recurrent with $\mathbb{P}\left(\tau_{x,0} = \infty\right) = 0$. Moreover, it is: null recurrent if $C_2 = \infty$, positive recurrent if $C_2 < \infty$.

Time to first extinction starting from $x \geq 1$

Let $\tau_{x,0}$ be the time it takes to first hit 0, starting from x. Let $\phi_{x,0}(z) = \mathbb{E}\left(z^{\tau_{x,0}}\right)$. With $\phi(z) = (\phi_{1,0}(z), \phi_{2,0}(z), \ldots)'$ the column-vector of the $\phi_{x,0}(z)$, and $\mathbf{q} = (q_1, q_2, \ldots)'$ the column-vector of the q_x, $\phi(z)$ solves:

$$\phi(z) = z\mathbf{q} + z\overline{P}\phi(z), \tag{7}$$

whose formal solution is $\phi(z) = z\left(I - z\overline{P}\right)^{-1}\mathbf{q}$, involving the resolvent of $\overline{P}$ which is computable. We get

$$\phi_{x,0}(z) = \sum_{y \geq x} q_y z^{y-x+1} \prod_{y'=x}^{y-1} p_{y'}.$$

Equivalently, with $x \geq 1$, the pmf of $\tau_{x,0}$ reads

$$\mathbb{P}(\tau_{x,0} = k) = q_{k+x-1} \prod_{y'=x}^{k+x-2} p_{y'},\ k \geq 1.$$

First-passage and Green kernel

Generating function of $P^n(x,y)$ (the Green potential function of the chain):

$$g_{x,y}(z) \equiv \sum_{n=0}^{\infty} z^n \mathbb{P}_x(X_n = y) = \sum_{n=0}^{\infty} z^n P^n(x,y) = (I - zP)^{-1}(x,y)$$

$$\phi_{x,y}(z) = \mathbb{E}\left(z^{\tau_{x,y}}\right) = \frac{g_{x,y}(z)}{g_{y,y}(z)}.$$

$$g_{x,x}(z) = (I - zP)^{-1}(x,x) = \frac{1 - \sum_{x'=0}^{x-1} z^{x'+1} q_{x'} \prod_{y=1}^{x'-1} p_y}{1 - \sum_{x'\geq 0} z^{x'+1} q_{x'} \prod_{y=0}^{x'-1} p_y}$$

giving the Green kernel as

$$g_{x,y}(z) = g_{x,x}(z)\, z^{y-x} \prod_{y'=x}^{y-1} p_{y'} \text{ if } y \geq x$$

$$g_{x,y}(z) = \left(g_{x,x}(z) - 1\right) z^{y-x} / \prod_{y'=y}^{x-1} p_{y'} \text{ if } 0 \leq y < x.$$

$$g_{0,0}(z) = 1/\left(1 - \phi_{0,0}(z)\right) = \sum_{n=0}^{\infty} z^n \mathbb{P}_0(X_n = 0),$$

where $\mathbb{P}_0(X_n = 0)$ is the contact probability at 0 at time n.

Singularity analysis and contact probability at 0: critical case

- If $0 < \alpha < 1$, in the null recurrent case (algebraic decay of the contact probability)

$$\mathbb{P}_0 \left(X_n = 0 \right) \underset{n \uparrow \infty}{\sim} \frac{\Gamma \left(\nu + 1 - \alpha \right)}{p_0 \Gamma \left(\nu + 1 \right) \Gamma \left(1 - \alpha \right) \Gamma \left(\alpha \right)} n^{-(1-\alpha)}.$$

When $\alpha = 1 - \varepsilon$ ($\varepsilon > 0$ small), the constant in front of $n^{-(1-\alpha)}$ vanishes like $\varepsilon / \left(p_0 \nu \right)$.
- When $\alpha = 1$, logarithmic singularity

$$\mathbb{P}_0 \left(X_n = 0 \right) \underset{n \uparrow \infty}{\sim} \frac{1}{p_0 \nu \log n}.$$

- When $\alpha > 1$ (positive recurrence case), $\mathbb{P}_0 \left(X_n = 0 \right) \underset{n \uparrow \infty}{\rightarrow} \pi_0 = 1 / \left(1 + p_0 \frac{\nu}{\alpha - 1} \right)$.
When $\alpha = 1 + \varepsilon$ ($\varepsilon > 0$ small): $\pi_0 \sim \varepsilon / \left(p_0 \nu \right)$, just like when $\alpha < 1$.

Total disaster: continuous-time

Invariant measure modified by the adjunction of holding rates.

$$\overline{\pi}_x = \overline{\pi}_0 r_0 \frac{\prod_{y=0}^{x-1} p_y}{r_x},\ x \geq 0.$$

Time change leading from X to $\overline{X}$ does not change road map of X so recurrence criterion is identical for both X and $\overline{X}$.

- $C_1 < \infty$ iff $\beta > 1$: CT MC is transient with no invariant measure.
- If $\beta < 1$, then $\overline{C}_2 \equiv \sum_{x \geq 1} \left(\frac{r_x}{r_0}\right)^{-1} \prod_{y=0}^{x-1} p_y < \infty$: MC is positive recurrent. For large x:

$$\overline{\pi}_x \sim x^{-\lambda} \prod_{y=0}^{x-1} \left(1 - \frac{\alpha}{\nu + y^{\beta}}\right) \sim x^{-\lambda} e^{-\alpha \sum^{x} y^{-\beta}} \sim x^{-\lambda} e^{-\frac{\alpha}{1-\beta} x^{1-\beta}}$$

with stretched exponential behaviour.

- In the critical case $\beta = 1$:

$$\overline{\pi}_x \sim x^{-\lambda} \prod_{y=0}^{x-1} \left(1 - \frac{\alpha}{\nu + y}\right) \sim x^{-(\lambda+\alpha)}$$

with power-law$(\alpha + \lambda)$ behaviour. The chain is recurrent. CT chain is positive recurrent if $\alpha + \lambda > 1$, null recurrent if $\alpha + \lambda \leq 1$. The chain is always positive recurrent if $\lambda > 1$.

A Pareto-Zipf variant of the model $(\alpha, \beta > 0)$

- given $X_n = x \in \{1, 2, ...\}$, the increment of X_n is

$$\begin{aligned} &+1 \text{ with probability}: \; p_x = \left(1 + x^{-\beta}\right)^{-\alpha} \\ &-x \text{ with probability}: \; q_x = 1 - \left(1 + x^{-\beta}\right)^{-\alpha}. \end{aligned} \tag{8}$$

- given $X_n = 0$, increment of X_n is $+1$ with prob. $p_0 \leq 1$ and 0 with prob. $q_0 = 1 - p_0$.

- $C_1 = \sum_{y=1}^{\infty} q_y < \infty$ iff $\beta > 1$: MC is transient with no invariant measure.
- If $\beta < 1$, then $C_1 = \infty$ and $C_2 = \sum_{x \geq 1} \prod_{y=0}^{x-1} p_y < \infty$: the MC is positive recurrent. Furthermore, for large x:

$$\pi_x \sim \prod_{y=0}^{x-1} \left(1 + y^{-\beta}\right)^{-\alpha} \sim e^{-\alpha \sum^x y^{-\beta}} \sim e^{-\frac{\alpha}{1-\beta} x^{1-\beta}}$$

with stretched exponential behaviour.
- In the critical case $\beta = 1$:

$$\pi_x \sim \prod_{y=0}^{x-1} \left(1 + y^{-1}\right)^{-\alpha} \sim e^{-\alpha \sum^x y^{-1}} \sim x^{-\alpha}$$

with power-law(α) behaviour. Chain is recurrent, positive recurrent if $\alpha > 1$, null recurrent if $\alpha \leq 1$.

Return to 0

- If $\beta > 1$: the chain is transient with

$$\mathbb{P}\left(\tau_{0,0} = \infty\right) = \prod_{y=0}^{\infty}\left(1 + y^{-\beta}\right)^{-\alpha} > 0.$$

- If $\beta < 1$: the chain is positive recurrent

$$\mu := \mathbb{E}\left(\tau_{0,0}\right) = 1 + C_2 = 1 + \sum_{x\geq 1}\prod_{y=0}^{x-1}\left(1 + y^{-\beta}\right)^{-\alpha} < \infty.$$

- Critical case: $\beta = 1$, $\alpha > 0$. In this case, let

$$\begin{aligned}\psi_0\left(z\right) \quad : \quad &= \sum_{x\geq 1} q_x z^x \prod_{y=1}^{x-1} p_y = \sum_{x\geq 1} z^x \left(1 - \left(1 + x^{-1}\right)^{-\alpha}\right)\prod_{y=1}^{x-1}\left(1 + y^{-1}\right)^{-\alpha} \\ &= \sum_{x\geq 1} z^x \left(1 - \left(1 + x^{-1}\right)^{-\alpha}\right) x^{-\alpha} = 1 - \frac{1}{z}\left(1 - z\right)\mathrm{Li}_{\alpha}\left(z\right)\end{aligned}$$

involving the polylog function: $\mathrm{Li}_{\alpha}\left(z\right) := \sum_{x\geq 1} z^x x^{-\alpha}$.
$\psi_0\left(z\right)$ is pgf of a discrete Pareto rv $P \geq 1$ with tail parameter α. The rv P can be identified to $\tau_{0,0}^{+} - 1 \overset{d}{=} H^{+}$.

Return time to 0

$$\phi_{0,0}(z) = \sum_{x\geq 0} q_x z^{x+1} \prod_{y=0}^{x-1} p_y = z\left(q_0 + p_0\psi_0(z)\right)$$
$$= z\left(1 - \frac{p_0}{z}(1-z)\,\mathrm{Li}_\alpha(z)\right).$$

Mean persistence time

$$\phi'_{0,0}(1) = \mathbb{E}\left(\tau_{0,0}\right) := \mu = 1 + C_2 = \frac{1}{\pi_0} = 1 + p_0\zeta(\alpha).$$

It can be checked that if $\mathbb{E}\left(\tau_{0,0}\right)$ is to exist, then necessarily $\mathbb{E}\left(\tau_{0,0}\right) > 1 + p_0$.
* This condition forces $\alpha > 1$: the positive recurrence condition for critical MCC.
Renewal aspects: rv $\tau_{0,0}^{+} - 1$ (also the height H^{+} of a true excursion):

$$\tau_{0,0}^{+} - 1 \overset{d}{=} H^{+} = \inf\left(x \geq 1 : B_x(\alpha) = 1\right), \tag{9}$$

where $\left(B_x(\alpha);\ x \geq 1\right)$ is a sequence of independent Bernoulli rvs obeying $\mathbb{P}\left(B_x(\alpha) = 1\right) = 1 - \left(1 + x^{-1}\right)^{-\alpha} = q_x$. It is the Pareto rv P with pgf $\psi_0(z)$.
Positive recurrent case ($\beta = 1,\ \alpha > 1$): With $C_2^{+} = \sum_{x\geq 1}\prod_{y=1}^{x-1} p_y = C_2/p_0 = \zeta(\alpha) < \infty$

$$\mathbb{E}\left(\tau_{0,0}^{0}\right) = p_0/q_0 \text{ and } \mathbb{E}\left(\tau_{0,0}^{+}\right) =: \mu^{+} = 1 + C_2^{+} = 1 + \zeta(\alpha) > 2.$$

Contact probability at 0: critical case

- If $0 < \alpha < 1$, singularity analysis shows that, in the null recurrent case (algebraic decay of the contact probability)

$$\mathbb{P}_0 \left(X_n = 0 \right) \underset{n \uparrow \infty}{\sim} \frac{1}{p_0 \Gamma \left(1 - \alpha \right)} n^{-(1-\alpha)}.$$

When $\alpha = 1 - \varepsilon$ ($\varepsilon > 0$ small), the constant in front of $n^{-(1-\alpha)}$ vanishes like ε / p_0.

- When $\alpha = 1$, logarithmic singularity

$$\mathbb{P}_0 \left(X_n = 0 \right) \underset{n \uparrow \infty}{\sim} \frac{1}{p_0 \log n}.$$

- When $\alpha > 1$ (positive recurrence case), $\mathbb{P}_0 \left(X_n = 0 \right) \underset{n \uparrow \infty}{\to} \pi_0 = 1 / \left(1 + p_0 \zeta \left(\alpha \right) \right)$. When $\alpha = 1 + \varepsilon$ ($\varepsilon > 0$ small), using $\zeta \left(\alpha \right) \underset{\alpha \uparrow 1}{\sim} \varepsilon^{-1}$: $\pi_0 \sim \varepsilon / p_0$, just like when $\alpha < 1$.

Invariant measure, positive recurrence: Zipf law

$$\frac{\pi_x}{1-\pi_0} = \mathbb{P}\left(X_\infty = x \mid X_\infty \geq 1\right) = \frac{x^{-\alpha}}{\zeta\left(\alpha\right)},\ x \geq 1$$

$Y_\infty := X_\infty \mid X_\infty \geq 1 \overset{d}{\sim} \text{Zipf}(\alpha - 1)$ so with

$$\begin{aligned}
\mathbb{E}\left(z^{Y_\infty}\right) &= \text{Li}_\alpha\left(z\right)/\text{Li}_\alpha\left(1\right) := \psi_\infty\left(z\right) \\
\mathbb{E}\left(z^{X_\infty}\right) &= \pi_0 + \left(1-\pi_0\right)\psi_\infty\left(z\right) = \left(1 + p_0 \text{Li}_\alpha\left(z\right)\right)/\left(1 + p_0 \text{Li}_\alpha\left(1\right)\right).
\end{aligned}$$

X_∞ is the mixture of $Y_\infty \geq 1$ and Y_0 which is degenerate at 0.
The shifted Zipf rv $Y_\infty - 1 := \left(X_\infty - 1 \mid X_\infty \geq 1\right) \geq 0$,, with pmf $\zeta\left(\alpha\right)^{-1}\left(x+1\right)^{-\alpha}$, $x \geq 0$ and pgf $z^{-1}\psi_\infty\left(z\right)$, is discrete SD. It has tail index $\alpha - 1 > 0$, with finite mean if $\alpha > 2$.
π_x is unimodal with mode at the origin ($\pi_{x+1}/\pi_x = \left(1 + x^{-1}\right)^{-\alpha} < 1$, $x \geq 1$, $\pi_1/\pi_0 = p_0 < 1$). Observing next that $\pi_x^2 \leq \pi_{x-1}\pi_{x+1}$ for $x \geq 2$, a sufficient condition for X_∞ to be log-convex and so infinitely divisible is that

$$\mathbb{P}\left(X_\infty = 1\right)^2 \leq \mathbb{P}\left(X_\infty = 0\right)\mathbb{P}\left(X_\infty = 2\right) \text{ which is } p_0 \leq 2^{-\alpha}.$$

Related research: Growth/Collapse vs Decay/Surge

- GROWTH/COLLAPSE[1]

Deterministic population growth models with power-law rates $\alpha(x) = \alpha_1 x^a$, $\alpha_1 > 0$, can exhibit a large variety of behaviors, ranging from algebraic ($a < 1$), exponential ($a = 1$) to hyperexponential (finite time explosion if $a > 1$). The dynamics $\dot{x}_t = \alpha(x_t)$, $x_0 = x$ describe the growth for the size (or mass) $x_t(x)$ of some population at time $t \geq 0$ and with initial condition $x \geq 0$. Variants of this model are introduced allowing logarithmic, exp-algebraic or even doubly exponential growth with time. The possibility of immigration is also raised.

We study a semi-stochastic catastrophe version $X_t(x)$ of such models. Here, at some jump times, possibly governed by state-dependent rates, the size of the population shrinks by a random amount of its current size, an event possibly leading to instantaneous local (or total) extinction. A special separable shrinkage transition kernel is investigated in more detail, including the case of total disasters. Between the jump times, the new process grows, following the deterministic dynamics started at the newly reached state after each jump. We discuss the conditions under which such processes are either transient or recurrent (positive or null), the scale function playing a key role in this respect, together with the speed measure cancelling the Kolmogorov forward operator.

Related research: Growth/Collapse vs Decay/Surge c'td

The scale function is also used to compute, when relevant, the law of the height of excursions. The question of the finiteness of the time to extinction is investigated together (when finite), with the evaluation of the mean time to extinction, either local or global. Some information on the embedded chain to the PDMP is also required when dealing with the classification of states 0 and ∞ that we exhibit.

Keywords: Deterministic population growth, catastrophe, PDMP, recurrence/transience, scale function, height and length of excursions, speed measure, expected time to extinction, classification of boundary states.

• DECAY/SURGE[2]

If $\alpha(x) \rightarrow \overline{\alpha}(x) := -\alpha(x)$ where α is as above non-negative, the wild-type population size with dynamics $\dot{x}_t = -\alpha(x_t)$, $x_0 = x$ now shrinks as time passes by, starting from x. We study the fate of such declining populations bound to fade away, reinvigorated by random bursts or surges. Kind of time-reversed process of the Growth/Collapse model.

[1] On population growth with catastrophes. Authors: Branda Goncalves, Thierry Huillet, Eva Löcherbach, arXiv:2007.03277 [pdf, other] math.PR

[2] On decay/surge population models. Authors: Branda Goncalves, Thierry Huillet, Eva Löcherbach, in preparation.