Parameter estimation of discretely observed interacting particle systems

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joint work with
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Plan of this talk

1. Model
2. Problem
3. Assumptions
4. Examples
5. Main results & proofs
6. Open problems

1. Model

Applications in

- mathematical biology (large societies of animals, neural networks)
- social sciences (multiagents' opinion, consensus)
- finance (systemic risk)
- mean-field games (a large number of players)...

An interacting particle system:

$$(X_t^{1,N}, \dots, X_t^{N,N})_{t\in[0,T]},$$

which satisfies

$$\begin{cases} \mathrm{d}X_t^{i,N} = b(X_t^{i,N}, \mu_t^N, \theta_1)\mathrm{d}t + a(X_t^{i,N}, \mu_t^N, \theta_2)\mathrm{d}W_t^i, \quad i = 1, ..., N,\ t \in [0,T], \\ \mathcal{L}(X_0^{1,N}, \dots, X_0^{N,N}) = \mu_0 \times \cdots \times \mu_0, \end{cases}$$

where

- $W^i := (W_t^i)_{t\in[0,T]}$, $i = 1, \dots, N$, are independent standard Brownian motions,
- $X_0^{1,N}, \dots, X_0^{N,N}$ are independent of $W^1, \dots, W^N$,
- $\mu_t^N := N^{-1} \sum_{i=1}^N \delta_{X_t^{i,N}}$,
- $b : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \times \Theta_1 \to \mathbb{R}$, $a : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \times \Theta_2 \to \mathbb{R}$, where $\mathcal{P}_2(\mathbb{R})$ is the set of all probability measures on $\mathbb{R}$ with finite 2nd moments endowed with the Wasserstein 2-metric W_2*, $\Theta_j \subset \mathbb{R}^{p_j}$ is compact, convex for some $p_j \in \mathbb{N}$, $j = 1, 2$,
- $\theta_j \in \Theta_j^\circ$, $j = 1, 2$, are parameters of interest.

*$W_2(\mu,\nu) := \inf_{\pi\in\Pi(\mu,\nu)}(\int_{\mathbb{R}\times\mathbb{R}} |x-y|^2 \pi(\mathrm{d}x,\mathrm{d}y))^{1/2}$, where $\Pi(\mu,\nu)$ denotes the set of all probability measures on $\mathbb{R}\times\mathbb{R}$ with marginals μ, ν

The associated McKean-Vlasov SDE (introduced by McKean 1966):

$$\begin{cases} \mathrm{d}\bar{X}_t = b(\bar{X}_t, \bar{\mu}_t, \theta_1)\mathrm{d}t + a(\bar{X}_t, \bar{\mu}_t, \theta_2)\mathrm{d}W_t, \quad t \in [0, T], \\ \bar{\mu}_0 = \mu_0, \end{cases} \tag{1}$$

where

- W is a standard Brownian motion,
- $\bar{X}_0$ is independent of W,
- $\bar{\mu}_t := \mathcal{L}(\bar{X}_t)$.

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The propagation of chaos property: if $\bar{X}^i$ satisfies (1) with $W = W^i$, $\bar{X}_0 = X_0^{i,N}$, $i = 1, \dots, N$, under A1, A2 (to be stated later), then

$$\mathbb{E}[|X_t^{i,N} - \bar{X}_t^i|^2] \le CN^{-1/2}, \quad i = 1, \dots, N,\ t \in [0, T].$$

This and other probabilistic properties: Sznitman 1984, 1991, Méléard 1996, Fernandez, Méléard 1997, Tanaka, Hitsuda 1981, Malrieu 2001, Bolley, Guillin, Villani 2007, Cattiaux, Guillin, Malrieu 2008, ...

2. Problem

Statistical inference:

- parametric: Kasonga 1990, Löcherbach 2002, Bishwal 2011, Giesecke, Schwenkler, Sirignano 2020, Genon-Catalot, Larédo 2021a, 2021b, 2022+, Chen 2021, Sharrock, Kantas, Parpas, Pavliotis 2021+, Della Maestra, Hoffmann 2022+
- nonparametric: Della Maestra, Hoffmann 2022, Belomestny, Pilipauskaitė, Podolskij 2022+, Belomestny, Podolskij, Zhou 2022+

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Problem of this talk: estimation of the parameter $\theta := (\theta_1, \theta_2)$ from observations

$$X^{i,N}_{j\Delta_n}, \ i = 1, \dots, N, \ j = 1, \dots, n,$$

with $\Delta_n := T/n$, where T is fixed and $N, n \to \infty$.

We define an estimator of $\theta = \theta_0$ as

$$\hat{\theta}_n^N \in \arg\min_{\theta \in \Theta} S_n^N(\theta),$$

where

$$S_n^N(\theta) := \sum_{0<i\leq N} \sum_{0\leq j<n} \left\{ \frac{(X_{(j+1)\Delta_n}^{i,N} - X_{j\Delta_n}^{i,N} - b(X_{j\Delta_n}^{i,N}, \mu_{j\Delta_n}^N, \theta_1)\Delta_n)^2}{a^2(X_{j\Delta_n}^{i,N}, \mu_{j\Delta_n}^N, \theta_2)\Delta_n} + \log a^2(X_{j\Delta_n}^{i,N}, \mu_{j\Delta_n}^N, \theta_2) \right\}$$

uses a Gaussian quasi-likelihood which originates from the Euler scheme:

$$X_{(j+1)\Delta_n}^{i,N} \approx X_{j\Delta}^{i,N} + b(X_{j\Delta_n}^{i,N}, \mu_{j\Delta_n}^N, \theta_1)\Delta_n + a(X_{j\Delta_n}^{i,N}, \mu_{j\Delta_n}^N, \theta_2)(W_{(j+1)\Delta_n}^i - W_{j\Delta_n}^i).$$

The Euler scheme contrast in case of the classical SDE: Florens-Zmirou 1989, Yoshida 1992, Kessler 1997

3. Assumptions

A1 μ_0 has finite k-th moment for all $k \in \mathbb{N}$.

A2 For all θ there exists a $C > 0$ such that all $(x, \mu), (y, \nu)$,

$$|b(x,\mu,\theta_1) - b(y,\nu,\theta_1)| + |a(x,\mu,\theta_2) - a(y,\nu,\theta_2)| \leq C(|x-y| + W_2(\mu,\nu)).$$

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A3 There exists a $C > 0$ such that $a^2(x, \mu, \theta_2) > C$ for all (x, μ, θ_2).

A4 (i) $b(x, \mu, \cdot) \in C^3(\Theta_1)$, $a(x, \mu, \cdot) \in C^3(\Theta_2)$ for all (x, μ) and their partial derivatives up to order 3 have polynomial growth in (x, μ) uniformly in θ.

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(ii) For all θ there exist $C > 0$, $k, l = 0, 1, \ldots$ such that for all $r_1 + r_2 = 1, 2$, $h_1, h_2 = 1, ..., p_1$, $\tilde{h}_1, \tilde{h}_2 = 1, ..., p_2$ and $(x, \mu), (y, \nu)$,

$$\begin{aligned} &|\partial^{r_1}_{\theta_{1,h_1}} \partial^{r_2}_{\theta_{1,h_2}} b(x,\mu,\theta_1) - \partial^{r_1}_{\theta_{1,h_1}} \partial^{r_2}_{\theta_{1,h_2}} b(y,\nu,\theta_1)| \\ &+|\partial^{r_1}_{\theta_{2,\tilde{h}_1}} \partial^{r_2}_{\theta_{2,\tilde{h}_2}} a(x,\mu,\theta_2) - \partial^{r_1}_{\theta_{2,\tilde{h}_1}} \partial^{r_2}_{\theta_{2,\tilde{h}_2}} a(y,\nu,\theta_2)| \\ &\leq C(|x-y| + W_2(\mu,\nu))(1+|x|^k+|y|^k+W_2^l(\mu,\delta_0)+W_2^l(\nu,\delta_0)). \end{aligned}$$

A5 For all $\varepsilon > 0$,

$$\inf_{\theta \in \Theta : \|\theta_1 - \theta_{0,1}\| \geq \varepsilon} I(\theta) > 0 \qquad \text{and} \qquad \inf_{\theta_2 \in \Theta_2 : \|\theta_2 - \theta_{0,2}\| \geq \varepsilon} J(\theta_2) > J(\theta_{0,2}),$$

where $I : \Theta \to \mathbb{R}$, $J : \Theta_2 \to \mathbb{R}$ are defined as

$$I(\theta) := \int_0^T \int_{\mathbb{R}} \frac{(b(x, \bar{\mu}_t, \theta_1) - b(x, \bar{\mu}_t, \theta_{0,1}))^2}{a^2(x, \bar{\mu}_t, \theta_2)} \bar{\mu}_t(\mathrm{d}x)\mathrm{d}t,$$

$$J(\theta_2) := \int_0^T \int_{\mathbb{R}} \left(\frac{a^2(x, \bar{\mu}_t, \theta_{0,2})}{a^2(x, \bar{\mu}_t, \theta_2)} + \log a^2(x, \bar{\mu}_t, \theta_2) \right) \bar{\mu}_t(\mathrm{d}x)\mathrm{d}t.$$

A6 The main-diagonal block elements of $\Sigma(\theta_0) := \mathrm{diag}(\Sigma^{(1)}(\theta_0), \Sigma^{(2)}(\theta_0))$ are $p_j \times p_j$ invertible $\Sigma^{(j)}(\theta_0) = (\Sigma^{(j)}_{kl}(\theta_0))$, $j = 1, 2$, where

$$\Sigma^{(1)}_{kl}(\theta_0) := 2\int_0^T \int_{\mathbb{R}} \frac{\partial_{\theta_{1,k}} b(x, \bar{\mu}_t, \theta_{0,1})\, \partial_{\theta_{1,l}} b(x, \bar{\mu}_t, \theta_{0,1})}{a^2(x, \bar{\mu}_t, \theta_{0,2})} \bar{\mu}_t(\mathrm{d}x)\mathrm{d}t,$$

$$\Sigma^{(2)}_{kl}(\theta_0) := \int_0^T \int_{\mathbb{R}} \frac{\partial_{\theta_{2,k}} a^2(x, \bar{\mu}_t, \theta_{0,2})\, \partial_{\theta_{2,l}} a^2(x, \bar{\mu}_t, \theta_{0,2})}{a^4(x, \bar{\mu}_t, \theta_{0,2})} \bar{\mu}_t(\mathrm{d}x)\mathrm{d}t.$$

A7 There are some $\tilde{a}, K \in C^2(\mathbb{R}^2)$ whose partial derivatives of order 1, 2 have polynomial growth such that for all (x, μ),

$$a(x, \mu, \theta_{0,2}) := \tilde{a}\Big(x, \int_{\mathbb{R}} K(x, y)\mu(\mathrm{d}y)\Big).$$

4. Examples

(i) The Kuramoto model: N oscillators defined by N angles $X_t^{i,N}$ (defined modulo 2π) evolving over $[0, T]$ according to

$$\mathrm{d}X_t^{i,N} = -\frac{\theta_1}{N} \sum_{j=1}^{N} \sin(X_t^{i,N} - X_t^{j,N})\mathrm{d}t + \theta_2 \mathrm{d}W_t^i.$$

Here

$$b(x, \mu, \theta_1) = -\theta_1 \int_{\mathbb{R}} \sin(x - y)\mu(\mathrm{d}y), \qquad a(x, \mu, \theta_2) = \theta_2.$$

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$$b(x,\mu,\theta_1) = -\theta_1\int_{\mathbb{R}}\sin(x-y)\mu(\mathrm{d}y), \qquad a(x,\mu,\theta_2) = \theta_2.$$

(ii) A model for opinion dynamics:

$$\mathrm{d}X_t^{i,N} = -\frac{1}{N}\sum_{j=1}^{N}\varphi(|X_t^{i,N} - X_t^{j,N}|,\theta_1)(X_t^{i,N} - X_t^{j,N})\mathrm{d}t + \theta_2\mathrm{d}W_t^i,$$

where the influence function $\varphi(x,\theta_1)$, which acts on the “difference of opinions” between agents, is an infinitely differentiable approximation of $\theta_{1,1}\mathbf{1}_{[0,\theta_{1,2}]}(x)$, $x\in\mathbb{R}$.

(iii) The interacting particle system

$$\mathrm{d}X_t^{i,N} = \left(\theta_{1,1} + \frac{\theta_{1,2}}{N}\sum_{j=1}^{N} X_t^{j,N} - \theta_{1,3}X_t^{i,N}\right)\mathrm{d}t + \theta_2\sqrt{1+(X_t^{i,N})^2}\,\mathrm{d}W_t^i$$

for $\theta_{1,2} = 0$ reduces to N independent samples of a special case of the Pearson diffusion, which has applications in finance.

(iv) The mean field limit of

$$\mathrm{d}X_t^{i,N} = \left(\theta_{1,1}+\frac{\theta_{1,2}}{N}\sum_{j=1}^{N} X_t^{j,N}-\theta_{1,3}X_t^{i,N}\right)\mathrm{d}t+\left(\theta_{2,1}+\theta_{2,2}\sqrt{\frac{1}{N}\sum_{j=1}^{N}(X_t^{j,N})^2}\right)\mathrm{d}W_t^i$$

is a time-inhomogeneous Ornstein-Uhlenbeck process. See Kasonga 1990 for $\theta_{1,1} = \theta_{2,2} = 0$.

5. Main results & proofs

Theorem (Consistency)

Assume A1-A5 without A4(ii). Then

$$\hat{\theta}_n^N \xrightarrow{\mathbb{P}} \theta_0 \quad \text{as } N, n \to \infty.$$

Theorem (Asymptotic normality)

Assume A1-A7. If $N\Delta_n \to 0$ then

$$\left(\sqrt{N}(\hat{\theta}_{n,1}^N - \theta_{0,1}), \sqrt{N/\Delta_n}(\hat{\theta}_{n,2}^N - \theta_{0,2})\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, 2(\Sigma(\theta_0))^{-1}\right) \quad \text{as } N, n \to \infty.$$

Technical results: Assume A1-A2.

- Then for all $k \in \mathbb{N}$, $N \in \mathbb{N}$, $i = 1, \ldots, N$, $s, t \in [0, T]$ it holds

$$\sup_{t\in[0,T]} \mathbb{E}[|X_t^{i,N}|^k] \le C, \qquad \mathbb{E}[|X_t^{i,N} - X_s^{i,N}|^k] \le C|t-s|^{k/2}.$$

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- Moreover, assume that $f : \mathbb{R} \times \mathcal{P}_l(\mathbb{R}) \to \mathbb{R}$ satisfies for some $C > 0$, $k, l = 0, 1, \dots$ and all $(x, \mu), (y, \nu)$,

$$|f(x,\mu) - f(y,\nu)| \\ \le C(|x-y| + W_2(\mu,\nu))(1 + |x|^k + |y|^k + W_l^l(\mu,\delta_0) + W_l^l(\nu,\delta_0))$$

and $(x, t) \mapsto f(x, \bar{\mu}_t)$ is in $L^1(\mathbb{R} \times [0,T], \bar{\mu}_t(\mathrm{d}x)\mathrm{d}t)$. Then as $N, n \to \infty$,

$$\frac{\Delta_n}{N} \sum_{0<i\le N} \sum_{0\le j<n} f(X_{j\Delta_n}^{i,N}, \mu_{j\Delta_n}^N) \xrightarrow{\mathbb{P}} \int_0^T \int_{\mathbb{R}} f(x, \bar{\mu}_t)\bar{\mu}_t(\mathrm{d}x)\mathrm{d}t.$$

Proof of consistency:

- Assume A1-A4(i). Then

$$\sup_{\theta\in\Theta} \left|\frac{\Delta_n}{N} S_n^N(\theta) - J(\theta_2)\right| = o_{\mathbb{P}}(1),$$

$$\sup_{\theta\in\Theta} \left|\frac{1}{N}(S_n^N(\theta) - S_n^N(\theta_{0,1},\theta_2)) - I(\theta)\right| = o_{\mathbb{P}}(1). \quad (2)$$

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- A5 implies that for every $\varepsilon>0$ there exists $\eta>0$ such that

$$\{\|\hat{\theta}_{n,1}^N-\theta_{0,1}\|\geq\varepsilon\}\subseteq\{I(\hat{\theta}_n^N)>\eta\},$$

where the definition of $\hat{\theta}_n^N$ and (2) imply

$$I(\hat{\theta}_n^N)\leq I(\hat{\theta}_n^N)-\frac{1}{N}(S_n^N(\hat{\theta}_n^N)-S_n^N(\theta_{0,1},\hat{\theta}_n^N))=o_{\mathbb{P}}(1).$$

The proof is similar in the case of $\hat{\theta}_{n,2}^N$.

Proof of asymptotic normality: By mean value theorem and definition of $\hat{\theta}_n^N$,

$$-\nabla_\theta S_n^N(\theta_0) = (\hat{\theta}_n^N - \theta_0) \int_0^1 \nabla_\theta^2 S_n^N(\theta_0 + s(\hat{\theta}_n^N - \theta_0)) ds.$$

Furthermore, we introduce

$$M_n^N := \operatorname{diag}\left(N^{-\frac{1}{2}} \mathbf{1}_{p_1}, \left(\frac{\Delta_n}{N}\right)^{\frac{1}{2}} \mathbf{1}_{p_2}\right),$$
$$\Sigma_n^N(\theta) := M_n^N \nabla_\theta^2 S_n^N(\theta) M_n^N.$$

Steps:

- If $N\Delta_n \to 0$ then

$$\nabla_\theta S_n^N(\theta_0) M_n^N = \sum_{0<j\le n} \xi_{n,j} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 2\Sigma(\theta_0))$$

follows from

$$\sum_{0<j\le n} \mathbb{E}_{(j-1)\Delta_n}[\xi_{n,j}] \xrightarrow{\mathbb{P}} 0$$

and a central limit theorem for martingale difference triangular arrays (Theorem 3.2 in Hall, Heyde 1980):

$$\sum_{0<j\le n} \mathbb{E}_{(j-1)\Delta_n}[\xi_{n,j}(\xi_{n,j})^\top] \xrightarrow{\mathbb{P}} 2\Sigma(\theta_0), \qquad \sum_{0<j\le n} \mathbb{E}_{(j-1)\Delta_n}[|\xi_{n,j,h}|^4] \xrightarrow{\mathbb{P}} 0,$$

where $h = 1, \dots, p_1 + p_2$ and $\mathbb{E}_t[\cdot]$ denotes the conditional expectation with respect to $\sigma(\{X_0^{i,N}, (W_s^i)_{s\in[0,t]}, i = 1, \dots, N\})$.

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- $\Sigma_n^N(\theta_0) \xrightarrow{\mathbb{P}} \Sigma(\theta_0)$,

- $\sup_{s\in[0,1]} \|\Sigma_n^N(\theta_0 + s(\hat{\theta}_n^N - \theta_0)) - \Sigma_n^N(\theta_0)\| \xrightarrow{\mathbb{P}} 0$.

6. Open problems

- Efficiency
- To improve $N\Delta_n \to 0$
- $X_t^{i,N} \in \mathbb{R}^d$
- Presence of jumps
- Non or semi-parametric estimation

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