# Bootstrap for integer-valued GARCH processes

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# Presentation plan

1. Integer-valued GARCH

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2. Properties of an INGARCH process

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# Integer-valued GARCH

Classical GARCH (Bollerslev, 1986):

$$\begin{aligned} X_t &= \sigma_t \, \varepsilon_t, \\ \sigma_t^2 &= \omega + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2, \end{aligned}$$

where $(\varepsilon_t)_{t \in \mathbb{Z}}$ i.i.d., $E\varepsilon_t = 0$, $E\left[\varepsilon_t^2\right] = 1$.

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Integer-valued counterpart: Poisson-INGARCH

$$\begin{aligned} X_t \mid \mathcal{F}_{t-1} &\sim \mathrm{Pois}(\lambda_t), \\ \lambda_t &= \omega + \sum_{i=1}^{p} \alpha_i X_{t-i} + \sum_{j=1}^{q} \beta_j \lambda_{t-j}, \end{aligned}$$

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# Our framework: Nonlinear Poisson-INGARCH

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X_t \mid \mathcal{F}_{t-1} &\sim \mathrm{Pois}(\lambda_t), \\
\lambda_t &= f_{\theta_0}\big(X_{t-1}, \ldots, X_{t-p}, \lambda_{t-1}, \ldots, \lambda_{t-p}\big)
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$\exists c_1, \ldots, c_p, d_1, \ldots, d_q \geq 0$, $\sum_{i=1}^p c_i + \sum_{j=1}^q d_j < 1$:

$$\left| f_{\theta_0}(x_1, \ldots, x_p, \lambda_1, \ldots, \lambda_q) - f_{\theta_0}(x_1', \ldots, x_p', \lambda_1', \ldots, \lambda_q') \right| \leq \sum_{i=1}^p c_i |x_i - x_i'| + \sum_{j=1}^q d_j |\lambda_j - \lambda_j'|$$

# Properties of the INGARCH process

Define $Z_t := (X_t, \ldots, X_{t-p+1}, \lambda_t, \ldots, \lambda_{t-q+1})$.
Then $\mathbf{Z} = (Z_t)_{t \in \mathbb{Z}}$ is a time-homogeneous Markov process with state space $S := \mathbb{N}_0^p \times [0, \infty)^q$.

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Implications:

- $\mathbf{Z}$ has a unique stationary distribution
- $\left(X_t\right)_{t\in\mathbb{Z}}$ is absolutely regular ($\beta$-mixing)

# Coupling

Metric on $S$:

$$\Delta_{\gamma,\delta}\big((x_1,\ldots,x_p,\lambda_1,\ldots,\lambda_q),(x_1',\ldots,x_p',\lambda_1',\ldots,\lambda_q')\big)$$
$$:= \sum_{i=1}^{p}\gamma_i|x_i - x_i'| + \sum_{j=1}^{q}\delta_j|\lambda_j - \lambda_j'|$$

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**Proposition 1** (Markov kernel is contractive)

Suppose that Condition 1 is fulfilled. Then, for an appropriate choice of $\gamma_1,\ldots,\gamma_p,\delta_1,\ldots,\delta_q > 0$, $\kappa < 1$, there exist

$$Z \sim P_{\theta_0}^{Z_t|Z_{t-1}=z} \qquad \text{and} \qquad Z' \sim P_{\theta_0}^{Z_t|Z_{t-1}=z'}$$

on a probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{P})$ such that

$$\widetilde{E}\Delta_{\gamma,\delta}(Z,Z') \leq \kappa\,\Delta_{\gamma,\delta}(z,z'). \tag{2.1}$$

**Proof of Proposition 1** For $z = (x_1, \ldots, x_p, \lambda_1, \ldots, \lambda_q)$, $z' = (x'_1, \ldots, x'_p, \lambda'_1, \ldots, \lambda'_q)$, define

$$\begin{aligned} Z &:= (X, x_1, \ldots, x_{p-1}, \lambda, \lambda_1, \ldots, \lambda_{q-1}), \\ Z' &:= (X', x'_1, \ldots, x'_{p-1}, \lambda', \lambda'_1, \ldots, \lambda'_{q-1}), \end{aligned}$$

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where, according to the model equation,

$$\lambda = f_{\theta_0}(x_1, \ldots, x_p, \lambda_1, \ldots, \lambda_q), \qquad \lambda' = f_{\theta_0}(x'_1, \ldots, x'_p, \lambda'_1, \ldots, \lambda'_q).$$

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To generate $X \sim \text{Pois}(\lambda)$ and $X' \sim \text{Pois}(\lambda')$, take a Poisson process with unit intensity: $\big(N(u)\big)_{u \geq 0}$ and define

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Then, there exist $\gamma_1, \ldots, \gamma_p, \delta_1, \ldots, \delta_q > 0$, $\kappa < 1$ such that

$$\widetilde{E} \Delta_{\gamma,\delta}(Z, Z') \leq \ldots \leq \kappa \Delta_{\gamma,\delta}(z, z').$$

# Another representation of the contraction property

Kantorovich (Wasserstein $L^1$) distance:

$$\mathcal{K}(Q, Q') := \inf_{Z \sim Q,\, Z' \sim Q'} \widetilde{E} \Delta_{\gamma,\delta}(Z, Z'),$$

where $Z$ and $Z'$ are random variables with respective distributions $Q$ and $Q'$, both defined on a common probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$.

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**Proposition 2** Suppose that Condition 1 is fulfilled. Then, for arbitrary distributions $Q$, $Q'$ on $S$,

$$\mathcal{K}\big(Q\pi^Z, Q'\pi^Z\big) \leq \kappa\, \mathcal{K}\big(Q, Q'\big),$$

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**Proof** Follows from (2.1). □

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**Proof**

- $Q \mapsto Q\pi^Z$ contractive
- $\mathcal{P} := \left\{Q\colon\ Q \text{ probab. distr. }, \int \|x\|\, dQ(x) < \infty\right\}$ complete

Banach fixed point theorem: $\pi^Z$ admits a unique fixed point $Q_0$, i.e. $Q_0\pi^Z = Q_0$. □

# Absolute regularity of the count process

**Corollary 2**

Suppose that Condition 1 is fulfilled. Then $(X_t)_{t\in\mathbb{Z}}$ is absolutely regular ($\beta$-mixing),

$$\beta_X(n) \le C\,\rho^n \qquad \forall\, n \in \mathbb{N},$$

for some $C < \infty$, $\rho < 1$.

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**Proof**

$$\begin{aligned}
&\beta_X(n)\\
&\quad= \;\; E\Big[\sup_C \Big| P\big((X_n, X_{n+1}, \ldots) \in C \mid X_0, X_{-1}, \ldots\big) - P\big((X_n, X_{n+1}, \ldots) \in C\big)\Big|\Big]
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&= E\Big[\sup_C \Big|P\big((X_n, X_{n+1}, \ldots) \in C \mid Z_0\big) - P\big((X_n, X_{n+1}, \ldots) \in C\big)\Big|\Big].
\end{aligned}$$

# Proof of Corollary 2 (contd.)

If $(\widetilde{Z}_t)_{t\in\mathbb{N}_0}$ and $(\widetilde{Z}'_t)_{t\in\mathbb{N}_0}$ are two versions of the process $(Z_t)_{t\in\mathbb{N}_0}$ such that $\widetilde{Z}_0$ and $\widetilde{Z}'_0$ are independent, then

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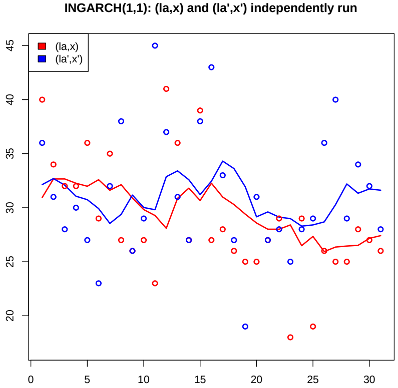
\leq\ & \sum_{l=0}^{\infty} \widetilde{P}\big(\widetilde{X}_{n+l} \neq \widetilde{X}'_{n+l}\big) \leq \sum_{l=0}^{\infty} \frac{1}{\gamma_1} \underbrace{\widetilde{E}\Delta_{\gamma,\delta}(\widetilde{Z}_{n+l}, \widetilde{Z}'_{n+l})}_{\leq C\rho^{n+l}} \leq \frac{C\,\rho^n}{\gamma_1\,(1-\rho)}
\end{aligned}
$$

□


INGARCH(1,1): (la,x) and (la',x') independently run
(la,x)
(la',x')
0
5
10
15
20
25
30
20
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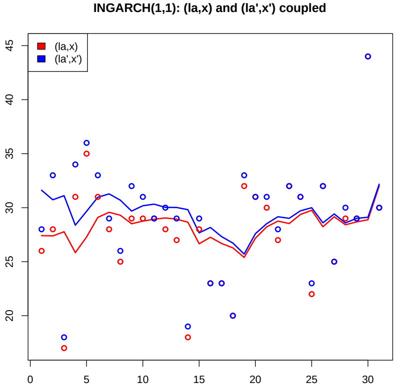
45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- Sometimes knowledge of the properties of $(X_t)_{t \in \mathbb{N}}$ required:
  - confidence intervals/sets (width)
  - hypothesis tests (critical value)
- "Bootstrap":
  Given $x_1, \ldots, x_n$, construct an artificial process $(X_t^*)_{t \in \mathbb{N}}$ which (hopefully) mimics the behavior of $(X_t)_{t \in \mathbb{N}}$

# Bootstrap (contd.)

In our case:

- parametric model:

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- estimate $\theta_0$ be $\widehat{\theta}_n = \widehat{\theta}_n(X_1, \ldots, X_n)$
  (e.g. by conditional maximum likelihood)

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$$

- hope for the best:

$$
P_{\theta_0}^{X_1^*, \ldots, X_n^* \mid X_1, \ldots, X_n} \quad \text{“}\approx\text{”} \quad P_{\theta_0}^{X_1, \ldots, X_n}
$$

# Bootstrap consistency

Usual approach:

- $S_n = S_n(X_1, \ldots, X_n; \theta_0)$ statistic of interest,
  e.g. $S_n = \sqrt{n}(\bar{X}_n - EX_1)$, $\bar{X}_n = n^{-1} \sum_{t=1}^n X_t$.
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More general approach:
Construct, on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$, respective versions $(\widetilde{Z}_t)_{t=1,\ldots,n}$ and $(\widetilde{Z}_t^*)_{t=1,\ldots,n}$ of $(Z_t)_{t=1,\ldots,n}$ and $(Z_t^*)_{t=1,\ldots,n}$ such that

$$\widetilde{E}\Delta_{\gamma,\delta}(\widetilde{Z}_t, \widetilde{Z}_t^*) \qquad \text{“\textit{small}”}$$

# Conditions for bootstrap consistency

**Condition 1'**

$\exists c_1, \ldots, c_p, d_1, \ldots, d_q \geq 0,\ \sum_{i=1}^{p} c_i + \sum_{j=1}^{q} d_j < 1$:

$$\left| f_\theta(x_1, \ldots, x_p, \lambda_1, \ldots, \lambda_p) - f_\theta(x_1', \ldots, x_q', \lambda_1', \ldots, \lambda_q') \right| \leq \sum_{i=1}^{p} c_i |x_i - x_i'| + \sum_{j=1}^{q} d_j |\lambda_j - \lambda_j'| \qquad \forall \theta \in \Theta_0,$$

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**Condition 3**

$$\left| f_\theta(x_1, \ldots, x_p, \lambda_1, \ldots, \lambda_q) - f_{\theta_0}(x_1, \ldots, x_p, \lambda_1, \ldots, \lambda_q) \right| \leq M \|\theta - \theta_0\| \Big( \sum_{i=1}^p x_i + \sum_{j=1}^q \lambda_j \Big),$$

for all $(x_1, \ldots, x_p, \lambda_1, \ldots, \lambda_q) \in S$ and all $\theta \in \Theta_0$.

# A characterization of bootstrap consistency

**Theorem 1**

Let Conditions 1', 2 and 3 be fulfilled. Then

$$\widetilde{E}\Delta_{\gamma,\delta}\big(\widetilde{Z}_t^*, \widetilde{Z}_t\big) \\ \leq \quad \frac{M\,E\|Z_0\|\,(\gamma_1+\delta_1)}{1-\kappa}\,\|\widehat{\theta}_n-\theta_0\| + \kappa^t\,\widetilde{E}\Delta_{\gamma,\delta}\big(\widetilde{Z}_0^*, \widetilde{Z}_0\big)$$

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For $t \geq 1$,

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where $(N_1(u))_{u\geq 0}$, $(N_2(u))_{u\geq 0}$, ... independent Poisson processes

# Properties of the coupled process

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Let Conditions 1', 2 and 3 be fulfilled. If $\widehat{\theta}_n \in \Theta_0$, then

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Draw on

$$\widetilde{E}\big[\Delta_{\gamma,\delta}(\widetilde{Z}_t, \widetilde{Z}_t') + \Delta_{\gamma,\delta}(\widetilde{Z}_t^*, \widetilde{Z}_t^{*\prime})\big] \le \kappa\, \widetilde{E}\big[\Delta_{\gamma,\delta}(\widetilde{Z}_{t-1}, \widetilde{Z}_{t-1}') + \Delta_{\gamma,\delta}(\widetilde{Z}_{t-1}^*, \widetilde{Z}_{t-1}^{*\prime})\big].$$

□

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- generalized means
- autocovariances
- degenerate von Mises and $U$-statistics

# Summary

- Poisson-INGARCH process:

$$X_t \mid \mathcal{F}_{t-1} \sim \mathrm{Pois}(\lambda_t), \quad \lambda_t = f_\theta\big(X_{t-1}, \ldots, X_{t-p}, \lambda_{t-1}, \ldots, \lambda_{t-q}\big)$$

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Details: Neumann, M. H. (2021). Bootstrap for integer-valued generalized GARCH(p,q) processes. *Statistica Neerlandica* **75** (3), 343–363.