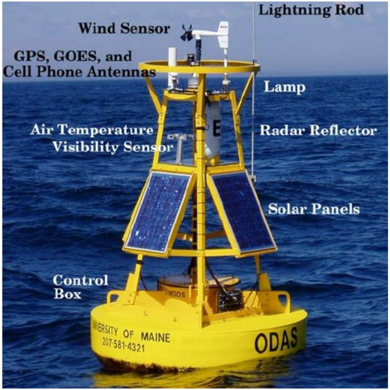
Inheritance of strong mixing and weak dependence under renewal sampling

Imma Valentina Curato
based on a joint work with D. Brandes and R. Stelzer

Institute of Mathematical Finance, University of Ulm

EcoDep 2021 Conference

The sentinel of the sea



Point Reference Data: sea surface temperature

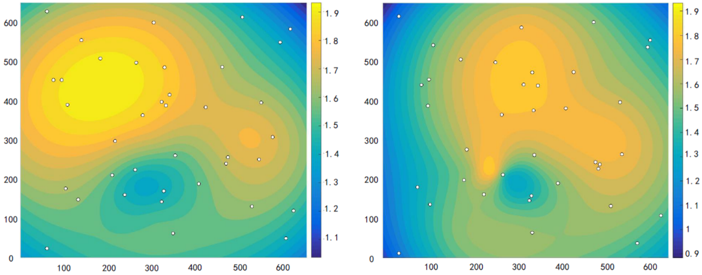


Figure: Spatio-Temporal reference point data in two time stamps. The white circles illustrates the locations of the sensors recording the temperature field values. The set $\{(t_s, x_s, Z_s)$ for $s = 1, \ldots, N\}$ is called point reference data set. Source: Wang et al. (2019), Deep learning for spatio-temporal data mining: a survey.

Following the reading of one sensor

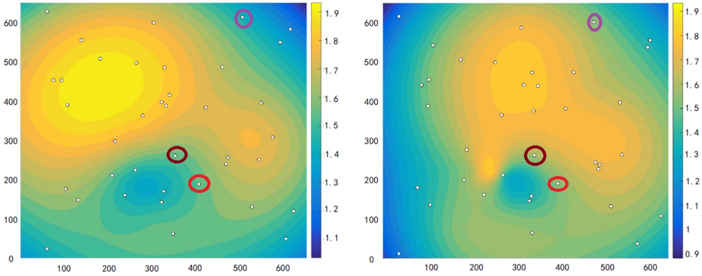
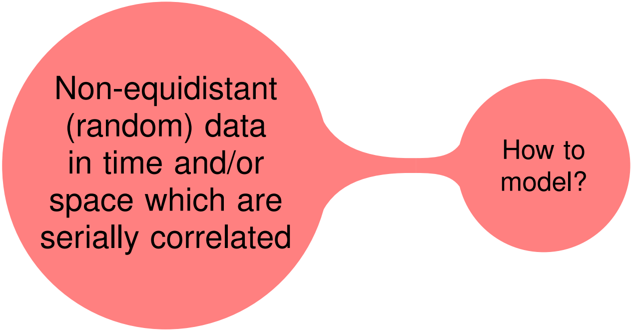


Figure: Trajectories data

Spatio-temporal trajectories data

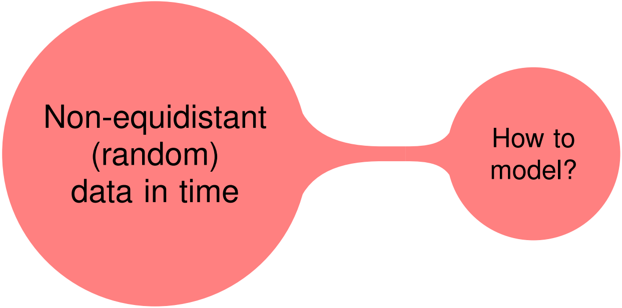


Time series

- Mobile, networked sensors can also be carried by people, (e.g., smartphones) or animals (e.g, animal tracking), enabling the monitoring of heart rate, body temperature, among other information.

Time series

- Mobile, networked sensors can also be carried by people, (e.g., smartphones) or animals (e.g, animal tracking), enabling the monitoring of heart rate, body temperature, among other information.



One framework

Let $\mathcal{I}$ denoting either $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Z}^m$ or $\mathbb{R}^m$

One framework

Let $\mathcal{I}$ denoting either $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Z}^m$ or $\mathbb{R}^m$

$$\downarrow$$

We study renewal sampling of $(X_t)_{t\in\mathcal{I}}$

Results

We show that if

Results

We show that if

- X is strictly-stationary, η, λ, κ, ζ, θ-weakly dependent, see Dedecker et al. (2008);

$$\downarrow$$

Results

We show that if

- X is strictly-stationary, η, λ, κ, ζ, θ-weakly dependent, see Dedecker et al. (2008);

$$\downarrow$$

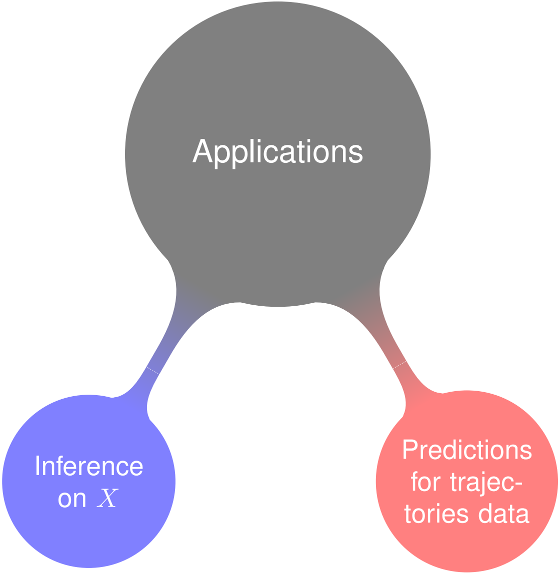
Results

We show that if

- X is strictly-stationary, η, λ, κ, ζ, θ-weakly dependent, see Dedecker et al. (2008);

$$\downarrow$$

- A renewal sampling of X is inheriting the dependence structure of X.


Applications
Inference on X
Predictions for trajectories data

Renewal Sampling

Let $\mathcal{I} \subseteq \mathbb{R}^m$ and $\tau = (\tau_i)_{i\in\mathbb{Z}\setminus\{0\}}$ be an $\mathcal{I}$-valued sequence of non-negative (component-wise) i.i.d. random vectors with distribution function μ such that $\mu\{0\} < 1$. For $i \in \mathbb{Z}$, we define an $\mathcal{I}$-valued stochastic process $(T_i)_{i\in\mathbb{Z}}$ as

$$T_0 := 0 \quad \text{and} \quad T_i := \begin{cases} \sum_{j=1}^{i} \tau_j\,, & i \in \mathbb{N}, \\ -\sum_{j=i}^{-1} \tau_j\,, & -i \in \mathbb{N}. \end{cases}$$

The sequence $(T_i)_{i\in\mathbb{Z}}$ is called a renewal sampling sequence.

Independence assumption

Independence assumption

TS: Observation times depend on the measuring instrument (typically sensors), i.e., on a random source independent of the process X, as observed by Bardet and Bertrand (2010).

ST: The sampling in space-time depends on the source of randomness proper of the instrument used to record them.

Renewal sampled process

Let $X = (X_t)_{t\in\mathcal{I}}$ and let $(T_i)_{i\in\mathbb{Z}}$ be a renewal sampling sequence independent of X. We define the sequence $Y = (Y_i)_{i\in\mathbb{Z}}$ as the stochastic process with values in $\mathbb{R}^{d+1}$ given by

$$Y_i = \begin{pmatrix} X_{T_i} \\ \tau_i \end{pmatrix}.$$

We call X the underlying process and Y the renewal sampled process.

Renewal sampled process

Let $X = (X_t)_{t \in \mathcal{I}}$ and let $(T_i)_{i \in \mathbb{Z}}$ be a renewal sampling sequence independent of X. We define the sequence $Y = (Y_i)_{i \in \mathbb{Z}}$ as the stochastic process with values in $\mathbb{R}^{d+1}$ given by

$$Y_i = \begin{pmatrix} X_{T_i} \\ \tau_i \end{pmatrix}.$$

We call X the underlying process and Y the renewal sampled process.

Remark: This modeling is designed to work when the sampling scheme is not known, i.e., it is not designed by an experimenter but just observed from the data.

Definition of Ψ-weak dependence

For any positive integer u, v, and functions F and G being bounded Lipschitz or bounded measurable functions, weakly dependent processes (or random fields) satisfy covariance inequalities of the following type:

$$\begin{aligned}|Cov(F(X_{i_1}, \ldots, X_{i_u}),&G(X_{j_1}, \ldots, X_{j_v}))| \\ &\leq c\,\Psi(\|F\|_\infty, \|G\|_\infty, Lip(F), Lip(G), u, v)\,\epsilon(r),\end{aligned} \tag{1}$$

where

Definition of Ψ-weak dependence

For any positive integer u, v, and functions F and G being bounded Lipschitz or bounded measurable functions, weakly dependent processes (or random fields) satisfy covariance inequalities of the following type:

$$
\begin{aligned}
&|Cov(F(X_{i_1}, \ldots, X_{i_u}), G(X_{j_1}, \ldots, X_{j_v}))| \\
&\qquad \leq c\,\Psi(\|F\|_\infty, \|G\|_\infty, Lip(F), Lip(G), u, v)\,\epsilon(r),
\end{aligned} \tag{1}
$$

where

- the sequence of coefficients $\epsilon = (\epsilon(r))_{r\in\mathbb{R}+}$ converges to zero as $r \to \infty$,
- c is a constant independent of r and $dist(\{i_1, \ldots, i_u\}, \{j_1, \ldots, j_v\}) \geq r$,
- the function $\Psi(\cdot)$ has different shapes depending on the functional spaces where F and G are defined, and the dependence notion under analysis.

Theorem (Brandes, C., Stelzer)

Let $Y = (Y_i)_{i\in\mathbb{Z}}$ be a $\mathbb{R}^{d+1}$-valued process with $X = (X_t)_{t\in\mathcal{I}}$ being strictly-stationary and Ψ-weakly dependent with coefficients $\epsilon = (\epsilon(r))_{r\in\mathbb{R}^+}$. Then, it exists a sequence $(\mathcal{E}(n))_{n\in\mathbb{N}^*}$ satisfying

$$|Cov(\tilde{F}(Y_{i_1},\ldots,Y_{i_u}),\tilde{G}(Y_{j_1},\ldots,Y_{j_v}))| \\ \leq C\,\Psi(\|\tilde{F}\|_\infty,\|\tilde{G}\|_\infty, Lip(\tilde{F}), Lip(\tilde{G}), u, v)\,\mathcal{E}(n)$$

where C is a constant independent of n, $dist(\{i_1,\ldots,i_u\},\{j_1,\ldots,j_v\}) \geq n$, and $\tilde{F}$, $\tilde{G}$ are either bounded Lipschitz or bounded measurable function. Moreover,

$$\mathcal{E}(n) = \int_{\mathcal{I}} \epsilon(\|r\|)\,\mu^{*n}(dr), \tag{2}$$

with μ^{*n} the n-fold convolution of μ.

If X is strictly stationary and η-weakly dependent

$$\begin{aligned}
&|Cov(\tilde{F}(Y_{i_1},\ldots,Y_{i_u}),\tilde{G}(Y_{j_1},\ldots,Y_{j_v}))| \\
&\qquad \leq C\, uLip(\tilde{F})\|\tilde{G}\|_\infty + vLip(\tilde{G})\|\tilde{F}\|_\infty\, \mathcal{E}(n)
\end{aligned}$$

Proposition (Brandes, C. and Stelzer)

- For F and G bounded measurable functions, and

$$\Psi(\|F\|_\infty, \|G\|_\infty, Lip(F), Lip(G), u, v) = \|F\|_\infty \|G\|_\infty$$

ϵ corresponds to the α**-coefficients** defined by Rosenblatt (1956).

Proposition (Brandes, C. and Stelzer)

- For F and G bounded measurable functions, and

$$\Psi(\|F\|_\infty, \|G\|_\infty, Lip(F), Lip(G), u, v) = \|F\|_\infty \|G\|_\infty$$

ϵ corresponds to the α-**coefficients** defined by Rosenblatt (1956).

Remark: we perform an alternative proof to the one of Charlot and Rachdi (2007).

θ-lex weak dependence, (C., Stelzer, and Ströh (2021))

- Lexicographic order on $\mathbb{R}^m$: for distinct elements $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$ and $z = (z_1, \ldots, z_m) \in \mathbb{R}^m$ we say $y <_{lex} z$ if and only if $y_1 < z_1$ or $y_p < z_p$ for some $p \in \{2, \ldots, m\}$ and $y_q = z_q$ for $q = 1, \ldots, p-1$.

θ-lex weak dependence, (C., Stelzer, and Ströh (2021))

- Lexicographic order on $\mathbb{R}^m$: for distinct elements $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$ and $z = (z_1, \ldots, z_m) \in \mathbb{R}^m$ we say $y <_{lex} z$ if and only if $y_1 < z_1$ or $y_p < z_p$ for some $p \in \{2, \ldots, m\}$ and $y_q = z_q$ for $q = 1, \ldots, p-1$.
- Let F be a bounded and G bounded Lipschitz functions, and $I_u = \{i_1, \ldots, i_u\} \subset \mathbb{R}^m$, and $j \in \mathbb{R}^m$ be such that $i_s <_{lex} j$ for all $s = 1, \ldots, u$, and $dist(I_u, j) \geq r$. Then,

$$\Psi(\|F\|_\infty, \|G\|_\infty, Lip(F), Lip(G), u, 1) = \|F\|_\infty Lip(G),$$

and ϵ corresponds to the **θ-lex-coefficients**.

Corollary (Brandes, C., Stelzer)

Let X be a strictly stationary and θ-lex weakly dependent random field defined on $\mathbb{R}^m$, and $\tau = (\tau_i)_{i \in \mathbb{Z} \setminus \{0\}}$ be an $\mathbb{R}^m$-valued sequence of non-negative (component-wise) i.i.d. random vector with distribution function μ. Then Y is a strictly stationary process, and there exists a sequence $\mathcal{E}$ such that

$$|Cov(\tilde{F}(Y_{i_1}, \dots, Y_{i_u}), \tilde{G}(Y_j))| \leq C \, \|F\|_\infty Lip(G) \, \mathcal{E}(n)$$

where C is a constant independent of n, and $\mathcal{E}$ are defined in (2).

Corollary (Brandes, C. and Stelzer)

If the coefficients $(\mathcal{E}(n))_{n\in\mathbb{N}}$ are finite, and converge to zero as n goes to infinity, then Y is Ψ-weakly dependent with coefficients $\mathcal{E}$.

Proposition (Brandes, C. and Stelzer)

- **Exponential Decay:** Let us assume that $\epsilon(r) \leq C\mathrm{e}^{-\gamma r}$ for $\gamma > 0$ and denote the Laplace transform of the distribution function μ of the inter-arrival time τ by

$$\mathcal{L}_\mu(t) = \int_{\mathbb{R}+} \mathrm{e}^{-tr}\,\mu(dr),\ t \in \mathbb{R}_+.$$

Then, the process Y admits coefficients

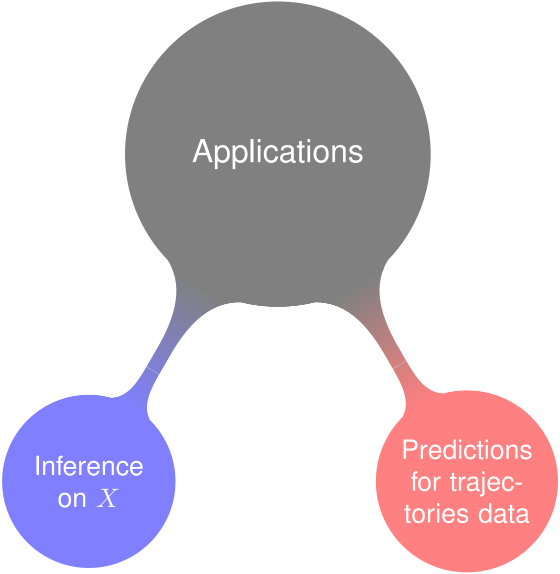
$$\mathcal{E}(n) \leq C\Big(\frac{1}{\mathcal{L}_\mu(\gamma)}\Big)^{-n}$$

which converge to zero as n goes to infinity.

Proposition (Brandes, C. and Stelzer)

- **Power decay:** Let us assume that $\epsilon(r) \leq Cr^{-\gamma}$ for $\gamma > 0$. Let $a > 0$ be a point in the support of the distribution function μ of the inter-arrival time τ such that $\mu([0, a)) > 0$, and set $p = \mu([a, \infty])$. Then, the process Y admits coefficients

$$\mathcal{E}(n) \leq C(nap)^{-\gamma} \quad \text{as } n \to \infty.$$


Applications
Inference on X
Predictions for trajectories data

Sufficient conditions

Given X centered, strictly stationary, and α-*strongly mixing* or *weakly dependent*:

Sufficient conditions

Given X centered, strictly stationary, and α-*strongly mixing* or *weakly dependent*:

- if $\mathbb{E}[\|X_0\|^\delta] < \infty$ for some $\delta > 0$;

Sufficient conditions

Given X centered, strictly stationary, and α-*strongly mixing* or *weakly dependent*:

- if $\mathbb{E}[\|X_0\|^\delta] < \infty$ for some $\delta > 0$;
- the weakly dependent or strongly mixing coefficients ϵ satisfy a condition

$$\sum_{i=1}^{\infty} \epsilon(n)^{A(\delta)} < \infty,$$

where $A(\delta)$ is a certain function of δ; then

Sufficient conditions

Given X centered, strictly stationary, and α-*strongly mixing* or *weakly dependent*:

- if $\mathbb{E}[\|X_0\|^{\delta}] < \infty$ for some $\delta > 0$;
- the weakly dependent or strongly mixing coefficients ϵ satisfy a condition
$$\sum_{i=1}^{\infty} \epsilon(n)^{A(\delta)} < \infty,$$
where $A(\delta)$ is a certain function of δ; then
- *central limit theorems* exist for a sample $(X_n)_{n\in\mathbb{Z}}$.

Sufficient conditions

Given X centered, strictly stationary, and α-*strongly mixing* or *weakly dependent*:

- if $\mathbb{E}[\|X_0\|^\delta] < \infty$ for some $\delta > 0$;
- the weakly dependent or strongly mixing coefficients ϵ satisfy a condition

$$\sum_{i=1}^{\infty} \epsilon(n)^{A(\delta)} < \infty,$$

where $A(\delta)$ is a certain function of δ; then

- *central limit theorems* exist for a sample $(X_n)_{n\in\mathbb{Z}}$.
- Knowing the dependence structure of Y enable us to check if the above condition hold for the sample $(Y_n)_{n\in\mathbb{Z}}$.

Trajectory data

- Let $\{(t_1^i, x_1^i), \dots, (t_p^i, x_p^i) : t_j^i \in \mathbb{R} \text{ and } x_j^i \in \mathbb{R}^2 \text{ for } i = 1, \dots, N \text{ and } j = 1, \dots, p\}$ represents a set of m different trajectories observed in p space-time points.
- $S = \{(X_1, Y_1), (X_2, Y_2), \dots, (X_N, Y_N)\}$ a generic *training set* where each *example* (X_i, Y_i) is determined by an input-output pair that we assume generated by a spatio-temporal random field $Z = (Z_t(x))_{(t,x)\in\mathbb{R}\times\mathbb{R}^2}$ such that

$$X_i = T_p(x_p^i), \quad \text{and} \quad Y_i = Z_{t_p^i}(x_p^i),$$

where $T_p(x_p^i) = (Z_{t_{p-1}^i}(x_{p-1}^i), \dots, Z_{t_1^i}(x_1^i))$ represents the past of the observation $Z_{t_p^i}(x_p^i)$ along the trajectory i.

Generalization bounds for trajectory data

Future Work:

- Linear predictors.
- Squared and absolute loss functions.

Thank you

BARDET, J.-M., AND BERTRAND, P. R.
A non parametric estimator of the spectral density of a continuous-time Gaussian process observed at random times.
Scand. J. Statist 37 (2010), 458–476.

BRANDES, D.-P., CURATO, I. V., AND STELZER, R.
Inheritance of strong mixing and weak dependence under renewal sampling.
arXiv:2007.00574v2, submitted for publication, 2020.

CHARLOT, F., AND RACHDI, M.
On the statistical properties of a stationary process sampled by a stationary point process.
Stat. Probabil. Lett. 78 (2008), 456–462.

CURATO, I. V., STELZER, R., AND STRÖH, B.
Central limit theorems for stationary random fields under weak dependence with application to ambit and mixed moving average fields.
Forthcoming in Annals of Applied Probability, arXiv:2007.10874v2 (2021).

DEDECKER, J., DOUKHAN, P., LANG, G., LEÓN, J. R., LOUHICHI, S., AND PRIEUR, C.
Weak dependence: with examples and applications.
Springer-Verlag, New York, 2007.

ROSENBLATT, M.
A central limit theorem and a strong mixing condition.
Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 43–47.

WANG, S. Z., CAO, J. N., AND YU, P. S.
Deep learning for spatio-temporal data mining: a survey.
arXiv: 1906.04928v2, 2019.