Taylor's Power Law and the Squared Coefficient of Variation

The (squared) coefficient of variation (CV) is defined as the sample variance over the square of the sample mean. The CV provides a measure of dispersion which is *invariant under scale change*. As such it is a useful complement to the variance. Given a set of temperature measurements on the Celsius scale the CV will be the same as the CV on the Fahrenheit scale. On the other hand, the sample variance will be different on the celsius scale than the variance on the Fahrenheit scale. This makes it difficult to interpret the results.

Taylor's Power Law provides a generalization of the CV. It is an important criterion used to describe the dynamic of populations. It was first observed as an empirical pattern in ecology where it was noticed that the variance of each sample was aproximately proportional to a power of its sample mean. An important property that Taylor's Law it is *invariant under scale change*. Taylor's law has been widely used as an explanation of numerous phenomena, including population growth in which samples at multiple locations at a point in time replicated at several points in time.

It has been used as a tool in studies in Demography, Biology and Physics, among others. Thousands of papers have been dedicated to the study of Taylor's Law. Important surveys on the topic are Kendal (2004), Eisler et al (2008) and Meng (2015). This talk is partly based partly in joint work with Mark Brown and Joel Cohen (2017).

Let $\{X_{i,j}\}, \quad i = 1, ..., n_j; \ n_j \geq 2, \quad j = 1, ..., m$ be a double array of non-negative random variables with $P(X_{i,j} = 0) = 0$. Let the sample mean and variance be denoted by $\bar{X}_{n,j} = \frac{\sum_{i=1}^{n_j} X_{i,j}}{n_j}$ $S^2_{n,j} = \frac{\sum_{i=1}^{n,j}(X_{i,j}-\bar{X}_j)^2}{n_j-1}$. Taylor's law is said to hold with exponent $\alpha > 0$ if $\log S^2_{n,j} \approx c + \alpha \log \bar{X}_{n,j}, \quad n_j \geq 2, \quad j = 1, ..., m$, for a fixed (finite) constant c. Rather than working with logarithms we study directly the behavior of the self-normalized random variable $\frac{S^2_{n,j}}{\bar{X}^{\alpha}_{n,j}}$.

Our main results are explicit formulas for $E\frac{S^2_{n,j}}{\bar{X}^{\alpha}_{n,j}}$, and $Var(\frac{S^2_{n,j}}{\bar{X}^{\alpha}_{n,j}})$ for all $j = 1, ..., m$ involving integrals of Laplace transforms of the distribution of the original variables. As an applications we obtain a measure of dispersion of the double array around the sample means. An illustrative example is given for the exponential distributions for which a simulation study shows that small sample sizes $n_j \leq 10$, $Var(\log(\frac{S^2_{n,j}}{\bar{X}^{\alpha}_{n,j}})) \geq Var(\frac{S^2_{n,j}}{\bar{X}^{\alpha}_{n,j}})$, further justifying the use of our approach to the study of Taylor's law.

We treat Taylor's Law as an inverse problem. That is, we say that Taylor's law holds with parameter α for all distributions $F_j, \;\; j = 1, ..., m$ for which

$$E(S_{n,j}^2 / \bar{X}_{n,j}^{\alpha}) := K(F, \alpha, n, j),$$

for all j. where $K(F, \alpha, n, j)$ is approximately a constant as n, j change. We present an example when the F_j's are exponential and more generally Gamma distributed. For this example Taylor's law holds for $\alpha = 2$.

A Reduction.

Without loss of generality we can drop the subscript j. Let $\{X_i, \ i = 1, ...n\}$ be a sequence of positive i.i.d. random variables with $P(X_i = 0) = 0$ and distribution function F. More precisely, by calculating

$$E\frac{S_n^2}{\bar{X}_n^\alpha} := K(F, \alpha, n), \tag{2}$$

in terms of the Laplace distribution of F. We then say that Taylor's Law holds with parameter $\alpha > 0$ for for all distributions for which $K(F, \alpha, n) \approx$ (a positive finite constant).

Back to the CV. Let $\mu \neq 0$ and $\sigma^2 < \infty$, be the mean and variance of X_1. It is common for $\frac{\sigma^2}{\mu^2}$ to be called the square coefficient of variation since it is variance of the random variable $\frac{X_1}{\mu}$ and it is unit-less. The natural estimator of this quantity (when $\bar{X}_n \neq 0$) is $\frac{S_n^2}{\bar{X}_n^2}$. It is easy to see that when the X's are non-negative the expectation is given by $K(F, 2, n)$, therefore connecting our results to the square coefficient of variation. However our results require that the X's be non-negative. A remarkable fact is that (because of cancellations) $K(F, 2, n)$ may be finite even when $\sigma^2 = \infty$.

Let $X, X_1, ..., X_n$ be a sequence of i.i.d. non-negative random variables. Let $\alpha > 0$ be a constant. in what follows, we provide a closed form expression for

$$E\frac{S_n^2}{\bar{X}_n^\alpha},$$

(when it is finite) in terms of functional of the Laplace transform, $\phi(\lambda)$, of distribution F_X. That is, $\phi(\lambda) = Ee^{-\lambda X}$ for all $\lambda > 0$.

Theorem.

$$E\frac{S_n^2}{\bar{X}_n^\alpha} = \frac{n^\alpha}{\Gamma(\alpha)} \int_0^\infty \lambda^{\alpha-1}\phi(\lambda)^{n-2}\{\phi''(\lambda)\phi(\lambda) - (\phi'(\lambda))^2\}d\lambda$$

for all fixed $\alpha > 0$.

Let us first recall that the properties of the Gamma density with fixed scale parameter α, shape parameter $\beta > 0$ implies that.

$$1 = \beta^{\alpha} \frac{1}{\Gamma(\alpha)} \int_{\lambda=0}^{\infty} \lambda^{\alpha-1} \exp\{-\lambda\beta\} d\lambda.$$

Remark 1.
Mean $= \mu = \frac{\alpha}{\beta}$
Variance $\sigma^2 = \frac{\alpha}{\beta^2}$
Coefficient of Variation $CV^2 = \frac{\sigma^2}{\mu^2} = \frac{1}{\alpha}$.
Special cases: Chi-square, exponential.

Lemma 1. Using the above, and letting $\beta = (\sum_{i=1}^{n} X_i)^{\alpha}$

$$\frac{1}{(\sum_{i=1}^{n} X_i)^{\alpha}} = \frac{1}{\Gamma(\alpha)} \int_{\lambda=0}^{\infty} \lambda^{\alpha-1} \exp\{-\lambda \sum_{i=1}^{n} X_i\} d\lambda.$$

Multiplying by X_1^2 on both sides and taking expectations we get,

$$E\frac{X_1^2}{(\sum_{i=1}^{n} X_i)^{\alpha}} = EX_1^2 \frac{1}{\Gamma(\alpha)} \int_{\lambda=0}^{\infty} \lambda^{\alpha-1} \exp\{-\lambda \sum_{i=1}^{n} X_i\} d\lambda =$$

$$\int_{\lambda=0}^{\infty} \frac{\lambda^{\alpha-1}}{\Gamma(\alpha)} EX_1^2 \exp\{-\lambda X_1\} E \exp\{-\lambda \sum_{i=2}^{n} X_i\} d\lambda =$$

(since the X's are i.i.d.)

$$\int_{\lambda=0}^{\infty} \frac{\lambda^{\alpha-1}}{\Gamma(\alpha)} EX_1^2 \exp\{-\lambda X_1\} \phi(\lambda)^{n-1} d\lambda =$$

$$\int_{\lambda=0}^{\infty} \frac{\lambda^{\alpha-1}}{\Gamma(\alpha)} \phi^{"}(\lambda) \phi(\lambda)^{n-1} d\lambda$$

$$(Since EX_1^2 \exp\{-\lambda X_1\} = \phi^{(}\lambda) \quad (by Tonellis)$$

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Lemma 2. Using the above, we have that

$$E\frac{X_1X_2}{(\sum_{i=1}^n X_i)^\alpha} = EX_1X_2\frac{1}{\Gamma(\alpha)}\int_{\lambda=0}^{\infty}\lambda^{\alpha-1}\exp\{-\lambda\sum_{i=1}^n X_i\}d\lambda =$$

$$\int_{\lambda=0}^{\infty}\frac{\lambda^{\alpha-1}}{\Gamma(\alpha)}EX_1\exp\{-\lambda X_1\}X_2\exp\{-\lambda X_2\}E\exp\{-\lambda\sum_{i=3}^n X_i\}d\lambda =$$

(by independence)

$$\int_{\lambda=0}^{\infty}\frac{\lambda^{\alpha-1}}{\Gamma(\alpha)}(\phi^{'}(\lambda))^2\phi(\lambda)^{n-2}d\lambda$$

(since $EX_1\exp\{-\lambda X_1\} = \phi'(\lambda)$ by Tonelli's)

Proof of Theorem 1: Since $S_n^2 = \frac{\sum_{1\le i<j\le n} \frac{(X_i-X_j)^2}{2}}{\binom{n}{2}}$

$$E\frac{S_n^2}{\bar{X}_n^\alpha} = \frac{n^\alpha}{\binom{n}{2}} \sum_{1\le i<j\le n} E\frac{\frac{(X_i-X_j)^2}{2}}{(\sum_{i=1}^n X_i)^\alpha} =$$

$$\frac{n^\alpha}{2} E\frac{(X_1-X_2)^2}{(\sum_{i=1}^n X_i)^\alpha} =$$

(by linearity of expectations and using the fact that the variables are i.i.d.)

$$n^\alpha\left(E\frac{X_1^2}{(\sum_{i=1}^n X_i)^\alpha} - E\frac{X_1X_2}{(\sum_{i=1}^n X_i)^\alpha}\right) =$$

$$\frac{n^\alpha}{\Gamma(\alpha)} \int_0^\infty \lambda^{\alpha-1}\phi(\lambda)^{n-2}\{\phi''(\lambda)\phi(\lambda) - (\phi'(\lambda))^2\}d\lambda,$$

by Lemma 1 and Lemma 2.

Examples.

In what follows we provide two examples using the exponential distribution and its relation to Taylor's law.

Example 1. For $\beta > 0$, let X_i be i.i.d. exponential, with $EX_1 = \frac{1}{\beta}$ then,

$$E\frac{S_n^2}{\bar{X}_n} = \frac{1}{\beta}\frac{n}{(n+1)}. \tag{3}$$

Since the right-hand side of (3) changes as β changes, Taylor's law does not hold for $\alpha = 1$.

In this case, $\phi(\lambda) = \frac{\beta}{\lambda+\beta}$, $\phi'(\lambda) = -\frac{\beta}{(\lambda+\beta)^2}$, $\phi''(\lambda) = \frac{2\beta}{(\lambda+\beta)^3}$, for all $\lambda > 0$. Therefore, by Theorem 1,

$$E\frac{S_n^2}{\bar{X}_n} = \frac{n}{\Gamma(1)}\int_0^\infty (\frac{\beta}{\lambda+\beta})^{n-2}[\frac{2\beta}{(\beta+\lambda)^3}\frac{\beta}{\beta+\lambda} - \frac{\beta^2}{(\beta+\lambda)^4}]d\lambda =$$

$$n\beta^n[0 - (-\frac{1}{(n+1)\beta^{n+1}}] = \frac{1}{\beta}\frac{n}{(n+1)}.$$

In conclusion, Taylor's law does not hold in our context when $\alpha = 1$.

In what follows we show that for Exponential random variables Taylor's law holds for $\alpha = 2$.

Example 2.

Let X be Gamma distributed with shape parameter 1 and scale parameter β (exponential wit $\mu = \frac{1}{\beta}$. In this case, we have that $\frac{\sigma^2}{\mu^2} = 1$ so the CV =1 and the traditional Taylor's law applies. A similar argument to the one given in (3) can be used to show that

$$E\frac{S_n^2}{\bar{X}_n^2} = \frac{n}{n+1},$$

independent of β showing that Taylor's law holds in our context when X_1 is exponential for all $\beta > 0$. In this case,

Bias $(CV^2) = \frac{n}{n+1} - 1 = -\frac{1}{n+1}$

Furtheremore, one also gets that

$$Var(\frac{S_n^2}{\bar{X}_n^2}) = \frac{4n^4}{(n-1)(n+1)^2(n+2)(n+3)} = \frac{4}{n} - \frac{24}{n^2} + O(\frac{1}{n^2})$$

Using Markov's inequality we obtain The following a concentration of measure result for Taylor's Law.

Proposition 1.

Let $\{X_{i,j},\ i = 1, ...n_j,\ j = 1, ..., m\}$ be a double array of non-negative random variables (m possible related sub-populations). Assume that for each j, $X_{i,j}$ $i = 1, ..., n_j$, is a sequence of identically distributed random variables with distribution F_j and that $P(X_{i,j} = 0) = 0$ for all i, j. Let $\bar{X}_{n,j} = \frac{\sum_{i=1}^{n_j} X_{i,j}}{n_j}$ be the sample mean and $S^2_{n,j} = \frac{\sum_{i=1}^{n,j}(X_{i,j} - \bar{X}_j)^2}{n_j - 1}$. the sample variance of these n_j variables. Let $W_{n,j} = \frac{S^2_{n,j}}{\bar{X}^{\alpha}_{n,j}}$) Then, for all $\epsilon > 0$,

$$P(\frac{\sum_{j=1}^{m}(W_{n,j} - EW_{n,j})^2}{m} \geq \epsilon) \leq \frac{1}{\epsilon^2 m^2} \sum_{j=1}^{m} Var(W_{n,j})$$

Example 3.

Assume that there is a group of (possibly related) m sub-populations and that each sub-population is comprised of $X_{i,j}$ $i = 1, ..., n_j$, identically distributed exponential (β_j) random variables.

Assume that $n = n_1 = n_2 = ... = n_m$ we have that the average squared deviation from of $W_{n,j}$ from its average satisfies

$$P(\frac{\sum_{j=1}^{m}(W_{n,j} - n/(n+1))^2}{m} \geq \epsilon) \leq \frac{1}{\epsilon^2 m}(\frac{4}{n} - \frac{24}{n^2} + O(\frac{1}{n^2})).$$

Remark.

Note that the upper-bound decreases linearly in both m and n. A simulation study gives the following results for exponential (β) random variables. ($Var(W_{n,j}), Var(\log(W_{n,j}))$, for

$n_j = 2, \quad (0.08, 3.973161)$
$n_j = 5, \quad (0.310019, 0.973161)$
$n_j = 10, \quad (0.2, .257)$
$n_j = 20, \quad (0.15, 1.449)$

Therefore, for small sample sizes our approach yields better approximations than the usual approach to Taylor's law.

On Taylor's Power Law of Fluctuation Scaling for Weakly Dependent Processes

In a forthcoming paper with Paul and Yahia we show that Taylor's Law with exponent 2 holds for a class of weakly dependent variables with an appropriately defined variance. In addition, we introduce a Central Limit Theorem of the self-normalized empirical version of Taylor's power Law in which the sample variance is divided by a power of the sample mean. Our frameworks is of interest in the context of ergodic Markov chains, infinite moving averages, ARCH and GARCH models where the family of distributions is in a parametric family. This self-normalised statistic is proved to be consistent. This consistency result is reinforced by the CLT. Both results together provide us with an asymptotic test of goodness-of-fit to check whether the corresponding dynamical Taylor's law indeed holds. Our work is mainly developed in the context of dynamical Taylor's law with exponent 2 but other powers less than 2 are also considered.

References

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