LECTURE 2: SOME MARKOV CHAINS ARISING IN POPULATION DYNAMICS

Abstract. **Keywords**: Population decay/growth, binomial/geometric catastrophe, recurrence/transience, height and length of excursions, invariant measure, time to extinction, Green kernel.

1. A generating function approach to binomial/geometric catastrophe Markov chains. Introduction

Besides systematic immigration events enhancing growth, populations are often subject to the effect of catastrophic events that cause recurrent mass removal. This results in a subtle balance between the two contradictory effects. We consider here the discrete-time Markovian evolution of a population whose size can either daily grow by a random amount or shrink by a fixed fraction of its current size. The latter event can be viewed as a strong size-dependent catastrophic event and the way it competes with the conflicting random growth events is analyzed. The questions of the existence and shape of an invariant probability measure, recurrence/transience criteria, time to local or global extinctions, Green kernel, are addressed in Section 2. The latter model is called the binomial catastrophe model, [19]. Some aspects of related models were recently addressed in [4], [1], [10], [2] and [15]. A variant of the binomial catastrophe model is also introduced and studied in section 3. Despite the apparent small changes in its definition, the impact on the asymptotic behavior is shown to be drastic.

Such binomial Markov chains (MCs) are random walks on the non-negative integers which differ from standard random walks on the integers in that a one-step move down from some positive integer cannot take one to a negative state so that the corresponding transition probabilities must be state-dependent.

2. The binomial catastrophe model

We first describe the model.

2.1. The model.

2.1. The model. Consider a discrete-time MC X_n taking values in $\mathbb{N}_0 := \{0,1,2,...\}$. With $b_n(c)$, $n = 1,2,...$ an independent identically distributed (iid) sequence of Bernoulli random variables (rvs) with success parameter c, let $c \circ X_n = \sum_{m=1}^{X_n} b_m(c)$ denote the Bernoulli thinning of X_n. Let β_n, $n = 1,2,...$ be an iid birth sequence of rvs with values in $\mathbb{N} = \{1,2,...\}$. The dynamics of the MC under concern here is a balance between birth and death events according as ($p + q = 1$):

$$\begin{aligned} X_{n+1} &= 1 \circ X_n + \beta_{n+1} = X_n + \beta_{n+1} \text{ wp } p \\ &= c \circ X_n \text{ wp } q = 1 - p. \end{aligned}$$

This model was considered by [19]. We have $c \circ X_n \sim$bin(X_n, c) hence the name binomial catastrophe. The binomial effect is appropriate when the catastrophe affects the individuals in a independent and even way, resulting in a drastic deletion of individuals at each step. Owing to: $c \circ X_n = X_n -$ $(1-c) \circ X_n$, the number of stepwise removed individuals is $(1-c) \circ X_n$ with probability (wp) q. This way of depleting the population size (at shrinkage times) by a fixed random fraction c of its current size is very drastic, especially if X_n happens to be large. Unless c is very close to 1 in which case depletion is modest (the case $c = 1$ is discussed below), it is very unlikely that the size of the upward moves will be large enough to compensate depletion while producing a transient chain drifting at ∞. We will make this very precise below.

Note also $X_n = 0 \Rightarrow X_{n+1} = \beta_{n+1}$ wp p, $X_{n+1} = 0$ wp q (reflection at 0 if $q > 0$).

Let $B_n(p)$, $n = 1, 2, ...$ be an iid sequence of Bernoulli rvs with $\mathbf{P}(B_1(p) = 1) = p$, $\mathbf{P}(B_1(p) = 0) = q$. Let $C_n(p, c) = B_n(p) + c\overline{B}_n(p) = c^{\overline{B}_n(p)}$ where $\overline{B}_n(p) = 1 - B_n(p)$. The above process' dynamics (driven by $B_n(p)\beta_n$) is compactly equivalent to

$$\begin{aligned} X_{n+1} &= c^{\overline{B}_{n+1}(p)} \circ X_n + B_{n+1}(p)\beta_{n+1},\ X_0 = 0 \\ &= \overline{B}_{n+1}(p) \cdot c \circ X_n + B_{n+1}(p)\left(X_n + \beta_{n+1}\right), \end{aligned}$$

$\left(X_n, \beta_{n+1}\right)$ being mutually independent. The thinning coefficients are now $c^{\overline{B}_{n+1}(p)}$, so random.

The one-step-transition matrix P of the MC X_n is: with $b_x := \mathbf{P}(\beta = x)$, $x \geq 1$ and $d_{x,y} := \binom{x}{y} c^y (1-c)^{x-y}$ the binomial probability mass function (pmf)

$$P(0,0) = q, P(0,y) = p\mathbf{P}(\beta = y) = pb_y, y \geq 1$$

$$P(x,y) = q\binom{x}{y} c^y (1-c)^{x-y} = qd_{x,y},\ x \geq 1 \text{ and } 0 \leq y \leq x \tag{1}$$

$$P(x,y) = p\mathbf{P}(\beta = y - x) = pb_{y-x},\ x \geq 1 \text{ and } y > x.$$

If β has first and second moment finite, with $\overline{c} = 1 - c$, $\overline{c} \circ x \sim$bin$(x, \overline{c})$, as x gets large

$$\begin{aligned} m_1(x) &= \mathbf{E}\left(\left(X_{n+1} - X_n\right) \mid X_n = x\right) = p\mathbf{E}(\beta) - q\overline{c}x \sim -q\overline{c}x \\ m_2(x) &= \mathbf{E}\left(\left(X_{n+1} - X_n\right)^2 \mid X_n = x\right) = p\mathbf{E}\left(\beta^2\right) + q\left(\overline{c}x + \overline{c}^2 x(x-1)\right) \sim q\overline{c}^2 x^2 \end{aligned}$$

with

$$\frac{m_1(x)}{m_2(x)} \sim -\frac{1}{\overline{c}x},\ x \text{ large.}$$

Note the variance of the increment is

$$\sigma^2\left(X_{n+1} - X_n \mid X_n = x\right) = p\sigma^2(\beta) + qc\overline{c}x \sim qc\overline{c}x.$$

2.2. **Special cases.** (i) When $c=1$, the lower triangular part of P vanishes leading to

$$\begin{aligned} P(0,0) &= q,\ P(0,y) = p\mathbf{P}(\beta = y) = pb_y,\ y \geq 1 \\ P(x,y) &= 0,\ x \geq 1 \text{ and } 0 \leq y < x,\ P(x,x) = q,\ x \geq 1 \\ P(x,y) &= p\mathbf{P}(\beta = y - x) = pb_{y-x},\ x \geq 1 \text{ and } y > x. \end{aligned}$$

The transition matrix P is upper-triangular with diagonal terms. The process X_n is non-decreasing, so it drifts to ∞.

(ii) When $c=0$ (total disasters),

$$\begin{aligned} P(0,0) &= q,\ P(0,y) = p\mathbf{P}(\beta = y) = pb_y,\ y \geq 1 \\ P(x,y) &= 0,\ x \geq 1 \text{ and } 0 < y \leq x,\ P(x,0) = q,\ x \geq 1 \\ P(x,y) &= p\mathbf{P}(\beta = y - x) = pb_{y-x},\ x \geq 1 \text{ and } y > x. \end{aligned}$$

When a downward move occurs, it takes instantaneously X_n to zero (a case of total disasters), independently of the value of X_n. This means that, defining $\tau_{x_0,0} = \inf(n \geq 1 : X_n = 0 \mid X_0 = x_0)$, the first extinction time of X_n, $\mathbf{P}(\tau_{x_0,0} = x) = qp^{x-1}$, $x \geq 1$, a geometric distribution with success parameter q, with mean $\mathbf{E}(\tau_{x_0,0}) = 1/q$, independently of $x_0 \geq 1$. Note that $\tau_{0,0}$, as the length of any excursion between consecutive visits to 0, also has a geometric distribution with success parameter q and finite mean $1/q$. In addition, the height H of an excursion is clearly distributed like $\sum_{x=1}^{\tau_{0,0}-1} \beta_x$ (with the convention $\sum_{x=1}^{0} \beta_x = 0$). Finally, this particular Markov chain clearly is always positive recurrent, whatever the distribution of β.

(iii) If $\beta \sim \delta_1$ a move up results in the addition of only one individual, which is the simplest deterministic drift upwards. In this case, the transition matrix P is lower-Hessenberg. This model constitutes a simple discrete version of a semi-stochastic growth/collapse model in the continuum.

2.3. **Recurrence versus transience.** Let $B(z) = \mathbf{E}\left(z^{\beta}\right)$ be the pgf of $\beta = \beta_1$, as an absolutely monotone function on $[0,1]$. Let $\boldsymbol{\pi}_n' = (\pi_n(0), \pi_n(1), \ldots)$ where $\pi_n(x) = \mathbf{P}_0(X_n = x)$. With $\mathbf{Z} = \left(1, z, z^2, \ldots\right)'$, a column vector obtained as the transpose $'$ of the row vector $\left(1, z, z^2, \ldots\right)$, define

$$\Phi_n(z) = \mathbf{E}_0\left(z^{X_n}\right) = \boldsymbol{\pi}_n' \mathbf{Z},$$

the pgf of X_n. With $D_{\mathbf{Z}} = \text{diag}\left(1, z, z^2, \ldots\right)$, $\mathbf{Z} = D_{\mathbf{Z}}\mathbf{1}$, the time evolution $\boldsymbol{\pi}_{n+1}' = \boldsymbol{\pi}_n' P$ yields

$$\Phi_{n+1}(z) = \boldsymbol{\pi}_{n+1}' \mathbf{Z} = \boldsymbol{\pi}_n' P \mathbf{Z} = \boldsymbol{\pi}_n' P D_{\mathbf{Z}} \mathbf{1},$$

leading to

$$\Phi_{n+1}(z) = pB(z)\,\Phi_n(z) + q\Phi_n(1 - c(1-z)),\ \Phi_0(z) = 1. \tag{2}$$

The fixed point pgf of X_∞, if it exists, solves

$$\Phi_\infty(z) = pB(z)\,\Phi_\infty(z) + q\Phi_\infty(1 - c(1-z)). \tag{3}$$

(i) When $c=1$, there is no move down possible. The only solution to $\Phi_\infty(z) = pB(z)\Phi_\infty(z) + q\Phi_\infty(z)$ is $\Phi_\infty(z)=0$, corresponding to $X_\infty \sim \delta_\infty$. Indeed, when $c=1$, combined to $\Phi_\infty(1)=1$,

$$\begin{aligned}\Phi_{n+1}(z) &= (q+pB(z))\,\Phi_n(z), \Phi_0(z)=1\\ \Phi_n(z)^{1/n} &= q+pB(z),\end{aligned}$$

showing that, if $1 \leq \rho := B'(1) = \mathbf{E}(\beta) < \infty$, $n^{-1}X_n \to q+p\rho \geq 1$ almost surely as $n\to\infty$. The process X_n is transient and drifts to ∞.

(ii) When $c=0$ (total disasters), combined to $\Phi_\infty(1)=1$

$$\Phi_\infty(z) = \frac{q}{1-pB(z)} =: \phi_\Delta(z) \tag{4}$$

is an admissible pgf solution. The law of X_∞ is a compound shifted-geometric sum Δ of the βs, whatever the distribution of β . Note that this is also the law of the height of any total disaster excursion, as the sample path between any two consecutive visits of X_n to 0. The length of such excursions clearly is geometric with success probability q.

2.4. **Existence and shape of invariant pmf ($c\in[0,1)$). • The case $c\in(0,1)$.** The limit law pgf $\Phi_\infty(z)$, if it exists, solves the functional equation

$$\Phi_\infty(z) = \phi_\Delta(z)\,\Phi_\infty(1-c(1-z)),$$

so that, formally

$$\Phi_\infty(z) = \prod_{n\geq 0}\phi_\Delta(1-c^n(1-z)), \tag{5}$$

an infinite product pgf.

Proposition. *The invariant measure exists for all $c\in(0,1)$ if and only if* $\mathbf{E}\left(\log_+\Delta\right) < \infty$.

Proof (Theorem 2 in [19]): By a comparison argument, we need to check the conditions under which $\pi(0) = \Phi_\infty(0)$ converges to a positive number.

$$\begin{aligned}\Phi_\infty(0) &= \prod_{n\geq 0}\phi_\Delta(1-c^n) > 0 \Leftrightarrow \sum_{n\geq 0}(1-\phi_\Delta(1-c^n)) < \infty\\ &\Leftrightarrow \int_0^1 \frac{1-\phi_\Delta(z)}{1-z}dz < \infty \Leftrightarrow \sum_{x\geq 1}\log x\mathbf{P}(\Delta = x) = \mathbf{E}\left(\log_+\Delta\right) < \infty,\end{aligned}$$

meaning that Δ has a finite logarithmic first moment.

For most βs therefore, the process X_n is positive recurrent, in particular if β has finite mean.

When β has finite first and second order moments, so do Δ and X_∞ which exist. Indeed:

If $B'(1) = \mathbf{E}\beta = \rho < \infty$, (with $\mathbf{E}\Delta = (p\rho)/q$)

$$\Phi'_{\infty}(1) = q\left(\frac{p\rho}{q^2} + \frac{c}{q}\Phi'_{\infty}(1)\right) \Rightarrow \Phi'_{\infty}(1) = \mathbf{E}(X_{\infty}) =: \mu = \frac{p\rho}{q(1-c)} < \infty$$

If $B''(1) < \infty$, X_{∞} has finite variance:

$$\Phi''_{\infty}(1) - \Phi'_{\infty}(1)^2 = \frac{p}{q}\frac{1}{1-c^2}\left(B''(1) + 2\frac{p}{q}\frac{2c-1}{1-c}\rho^2\right)$$

Counter-example: With $\beta, C > 0$, suppose that $\mathbf{P}(\beta > x) \sim_{x\uparrow\infty} C(\log x)^{-\beta}$ translating that β has very heavy logarithmic tails (any other than logarithmic slowly varying function would do the job as well). Then $\mathbf{E}\beta^q = \infty$ for all $q > 0$ and β has no moments of arbitrary positive order. Equivalently, $B(z) \sim_{z\downarrow 1} 1 - \frac{C}{(-\log(1-z))^{\beta}}$. Therefore, with $C' = pC/q$,

$$\phi_{\Delta}(z) = \frac{q}{1-pB(z)} \sim_{z\downarrow 1} 1 - \frac{C'}{(-\log(1-z))^{\beta}}$$

translating that $\mathbf{P}(\Delta > y) \sim_{y\uparrow\infty} C'(\log y)^{-\beta}$ shares the same tail behavior as β. From this, $\mathbf{P}(\log\Delta > x) \sim_{j\uparrow\infty} C'x^{-\beta}$ so that $\log\Delta$ has a first moment if and only if $\beta > 1$. For such a (logarithmic tail) model of β, we conclude that X remains positive recurrent if $\beta > 1$ and starts being transient only if $\beta < 1$. The case $\beta = 1$ is a critical null-recurrent situation. Being strongly attracted to 0, the binomial catastrophe model exhibits a recurrence/transience transition but only for such very heavy-tailed choices of β.

Corollary. *If the process is null recurrent or transient, no non-trivial ($\neq \mathbf{0}'$) invariant measure exists.*

Proof: This is because, $\Phi_{\infty}(z)$ being an absolutely monotone function on $[0,1]$ if it exists,

$$\pi(0) = \Phi_{\infty}(0) = 0 \Rightarrow \Phi_{\infty}(1-c) = 0 \Rightarrow \pi(x) = 0 \text{ for all } x \geq 1.$$

Clustering (sampling at times when thinning occurs, time change): Let $G = \inf(n \geq 1 : B_n(p) = 0)$, with $\mathbf{P}(G = k) = p^{k-1}q$, $\mathbf{E}\left(z^{G-1}\right) = \frac{q}{1-pz}$. G is the time elapsed between two consecutive catastrophic events. So long as there is no thinning of X (a catastrophic event), the process grows of $\Delta = \sum_{k=1}^{G-1}\beta_k$ individuals. Consider a time-changed process of X whereby one time unit is the time between consecutive catastrophic events. During this laps of time, the original process X_n grew of Δ individuals, before shrinking to a random amount of its current size at catastrophe times. We are thus led to consider the time-changed integral-valued Ornstein-Uhlenbeck process [also known as an INAR(1) process, see [18]]

$$X_{k+1} = c \circ X_k + \Delta_{k+1},\ X_0 = 0,$$

with Δ_k, $k = 1, 2, ...$ an iid sequence of compound shifted geometric rvs.

In this form, X_k is also a pure-death subcritical branching process with immigration, Δ_{k+1} being the number of immigrants at generation $k+1$, independent of X_k. With $\Phi_k(z) = \mathbf{E}\left(z^{X_k}\right)$, we have

$$\Phi_{k+1}(z) = \frac{q}{1 - pB(z)} \Phi_k(1 - c(1 - z)), \; \Phi_0(z) = 1.$$

The limit law (if it exists) $\Phi_\infty(z)$ also solves

$$\Phi_\infty(z) = \frac{q}{1 - pB(z)} \Phi_\infty(1 - c(1 - z)).$$

Thus

$$\Phi_\infty(z) = \prod_{n \geq 0} \phi_\Delta(1 - c^n(1 - z)),$$

corresponding to

$$X_\infty \stackrel{d}{=} \sum_{n \geq 0} c^n \circ \Delta_{n+1}.$$

As conventional wisdom suggests, the time-changed process has the same limit law as the original binomial catastrophe model, so if and only if the condition $\mathbf{E} \log_+ \Delta < \infty$ holds.

Proposition. *When the law of X_∞ exists, it is discrete self-decomposable (SD).*

Proof: This follows, for example, from Theorem 3.1 of [5] and the INAR(1) process representation. See [25] for an account on discrete SD distributions.

X_∞ being SD it is unimodal, with mode at the origin if $\pi(1) < \pi(0)$, or with two modes at $\{0, 1\}$ if $\frac{\pi(1)}{\pi(0)} = 1$ (see [26], Theorem 4.20).

With $\mathbf{P}(\Delta = 1) = \phi'_\Delta(0) = pq\mathbf{P}(\beta = 1)$, we have

$$\begin{aligned}
\pi(0) &= \Phi_\infty(0) = q\Phi_\infty(1 - c) \\
\pi(1) &= \Phi'_\infty(0) = \mathbf{P}(\Delta = 1)\Phi_\infty(1 - c) + qc\Phi'_\infty(1 - c) \\
&= \frac{\mathbf{P}(\Delta = 1)}{q}\pi(0) + qc\Phi'_\infty(1 - c) > p\mathbf{P}(\beta = 1)\pi(0)
\end{aligned}$$

A condition for unimodality at 0 is thus

$$(\log \Phi_\infty)'(1 - c) < \frac{1 - p\mathbf{P}(\beta = 1)}{c}. \tag{6}$$

Note also

$$\begin{aligned}
\pi(1) &= \Phi'_\infty(0) = \sum_{m \geq 0} c^m \phi'_\Delta(1 - c^m) \prod_{n \neq m} \phi_\Delta(1 - c^n) \\
&= \pi(0) \sum_{m \geq 0} c^m (\log \phi_\Delta)'(1 - c^m) \\
&= \pi(0) \sum_{m \geq 0} c^m \frac{pB'(1 - c^m)}{1 - pB(1 - c^m)}
\end{aligned}$$

giving a closed-form condition for unimodality at 0. For instance, if $B(z) = z$, $\pi(1) < \pi(0)$ if and only if

$$\sum_{m\geq 0} \frac{pc^m}{q+pc^m} < 1.$$

Tails of X_∞. The probabilities $\pi(x) = [z^x]\,\Phi_\infty(z)$, $x \geq 1$ (the z^x-coefficient in the power series expansion of of $\Phi_\infty(z)$) are hard to evaluate. However, some information on the large x tails $\sum_{y>x}\pi(y)$ can be estimated in some cases.

- Consider a case where $B(z) \sim C\cdot(1-z/z_c)^{-1}$ as $z \to z_c > 1$ so that $\mathbf{P}(\beta > x) \sim C\cdot z_c^{-x}$ has geometric tails. As detailed below,

$$\phi_\Delta(z) \sim C'\cdot(1-z/z_c')^{-1} \text{ as } z \to z_c' > 1$$

so that, with $C' = pqC/(1-pC) < 1$, $\mathbf{P}(\Delta > x) \sim C'\cdot z_c'^{-x}$ also has geometric (heavier) tails but with modified rate $z_c' = z_c(1-pC) < z_c$.

Then

$$\Phi_\infty(z) \sim C''\cdot(1-z/z_c')^{-1} \text{ as } z \to z_c' > 1$$

so that $\mathbf{P}(X_\infty > x) \sim C'\cdot z_c'^{-x}$ also has geometric tails with rate z_c'. Indeed, with $z_c'' = (z_c' - (1-c))/c > z_c'$,

$$\begin{aligned} C''\cdot(1-z/z_c')^{-1} &\sim C'C''\cdot(1-z/z_c')^{-1}\,(1-(1-c(1-z))/z_c')^{-1} \\ &= C'C''\left(\frac{z_c'}{cz_c''}\right)\cdot(1-z/z_c')^{-1}\,(1-z/z_c'')^{-1} \end{aligned}$$

showing that

$$\Phi_\infty(z) \sim C'C''\left(\frac{z_c'}{cz_c''}\right)(1-z_c'/z_c'')^{-1}\cdot(1-z/z_c')^{-1}, \text{ as } z \to z_c' > 1.$$

- Consider a positive recurrent case with $B'(1) = \mathbf{E}\beta = \infty$. This is the case if, with $\alpha \in (0,1)$, $B(z) \sim 1-(1-z)^\alpha$ as $z \to 1$, or if (unscaled Sibuya, see [24]) $B(z) = 1-(1-z)^\alpha$. In this case $\phi_\Delta(z) \sim 1-\frac{p}{q}(1-z)^\alpha$ scaled Sibuya (with scale factor $\frac{p}{q} = \mathbf{E}(G-1)$). Indeed,

$$\frac{q}{1-p(1-(1-z)^\alpha)} = \frac{1}{1+\frac{p}{q}(1-z)^\alpha} \underset{z\to 1}{\sim} 1-\frac{p}{q}(1-z)^\alpha$$

Then, X_n being recurrent, in view of $\Phi_\infty(z) = \frac{q}{1-pB(z)}\Phi_\infty(1-c(1-z))$,

$$\Phi_\infty(z) \underset{z\to 1}{\sim} 1-\gamma(1-z)^\alpha$$

where $\gamma = p/[q(1-c)^\alpha]$. Indeed, as $z \to 1$,

$$1-\gamma(1-z)^\alpha \sim \left[1-\frac{p}{q}(1-z)^\alpha\right][1-\gamma c^\alpha(1-z)^\alpha]$$

allowing to identify the scale parameter γ. The three rvs β, Δ and X_∞ have power law tails with index α.

• **The case $c = 0$ (total disasters)**.

In that case, combined to $\Phi_\infty(1) = 1$,

$$\Phi_\infty(z) = \frac{q}{1-pB(z)} = \phi_\Delta(z)$$

is an admissible pgf solution. The law of X_∞ is a compound shifted-geometric of the βs, whatever the distribution of β.

The probabilities $\pi(x) = [z^x]\,\Phi_\infty(z)$, $x \geq 1$ are explicitly given by the Faa di Bruno formula for compositions of two pgfs, [8]. Let us look at the tails of X_∞.

- If in particular, with $C \in (0,1)$, $B(z) \sim C \cdot (1 - z/z_c)^{-1}$ as $z \to z_c > 1$ so that $\mathbf{P}(\beta > x) \sim C \cdot z_c^{-x}$ has geometric tails, then

$$\Phi_\infty(z) \sim C' \cdot (1 - z/z_c')^{-1} \text{ as } z \to z_c' > 1$$

so that, with $C' = pqC/(1 - pC) < 1$, $\mathbf{P}(X_\infty > x) \sim C' \cdot z_c'^{-x}$ also has geometric (heavier) tails but with modified rate $z_c' = z_c(1 - pC) < z_c$.

- If $B(z) = \left(e^{\theta z} - 1\right)/\left(e^{\theta} - 1\right)$ (β is Poisson conditioned to be positive) is entire. $\Phi_\infty(z)$ has a simple pole at $z_c > 1$ defined by

$$\left(e^{\theta z_c} - 1\right)/\left(e^{\theta} - 1\right) = 1/p$$

and X_∞ has geometric tails with rate z_c.

- If, with $\alpha \in (0,1)$, $B(z) \sim 1 - (1-z)^\alpha$ as $z \to 1$, or if (unscaled Sibuya) $B(z) = 1 - (1-z)^\alpha$, then $\Phi_\infty(z) \sim 1 - \frac{p}{q}(1-z)^\alpha$ scaled Sibuya (with scale factor $\frac{p}{q} = \mathbf{E}(G-1)$:

$$\frac{q}{1 - p\left(1 - (1-z)^\alpha\right)} = \frac{1}{1 + \frac{p}{q}(1-z)^\alpha} \underset{z \to 1}{\sim} 1 - \frac{p}{q}(1-z)^\alpha$$

- Suppose, with $z_0 > 1$,

$$B(z) = \frac{1 - (1 - z/z_0)^\alpha}{1 - (1 - 1/z_0)^\alpha}$$

If there exists $z_c > 1 : B(z_c) = 1/p$ else if $B(z_0) = 1/\left(1 - (1 - 1/z_0)^\alpha\right) > 1/p$ ($(1 - 1/z_0)^\alpha > q$ or $z_0 > 1/\left(1 - q^{1/\alpha}\right)$), then $\Phi_\infty(z)$ has a simple algebraic pole at z_c. If no such z_c exists, $\Phi_\infty(z)$ is entire. This is reminiscent of a condensation phenomenon.

We finally observe that

$$\frac{1 - \phi_\Delta(z)}{1 - z} = \frac{p\,(1 - B(z))}{(1-z)(1 - pB(z))} = \frac{p}{q}\frac{1 - B(z)}{1 - z}\phi_\Delta(z)$$

$$\begin{aligned}
\mathbf{P}(\Delta > n) &= \frac{p}{q}\sum_{m=0}^{n}\mathbf{P}(\beta > n - m)\,\mathbf{P}(\Delta = m) \\
\mathbf{P}(\Delta = n) &= \frac{p}{q}\left(\sum_{m=0}^{n-1}\mathbf{P}(\beta = n - m)\,\mathbf{P}(\Delta = m) + \mathbf{P}(\Delta = n)\right)
\end{aligned}$$

When is Δ with $\phi_\Delta(z) = \frac{q}{1 - pB(z)}$ itself SD? In any case, Δ is at least infinitely divisible (ID) else compound Poisson because $\phi_\Delta(z) = \exp -r\,(1 - \psi(z))$ where $r > 0$ and $\psi(z)$ is a pgf with $\psi(0) = 0$. Indeed, with $q = e^{-r}$,

$$\psi(z) = \frac{-\log(1 - pB(z))}{-\log q}$$

is a pgf (the one of a Fisher-log-series rv).

Proposition. *With* $b_x = [z^x] B(z)$, $x \geq 1$, *the condition*

$$\frac{b_{x+1}}{b_x} \leq \frac{x - pb_1}{x+1} \text{ for any } x \geq 1$$

entails that Δ *is SD.*

Proof: If Δ is SD then (see [23], Lemma 2.13)

$$\phi_\Delta(z) = e^{-r\int_z^1 \frac{1-h(z')}{1-z'}dz'}$$

for some $r > 0$ and some pgf $h(z)$ obeying $h(0) = 0$. We are led to check if

$$\frac{pB'(z)}{1 - pB(z)} = r\frac{1 - h(z)}{1 - z}$$

for some pgf h and $r = pb_1$ where

$$h(z) = 1 - \frac{1}{b_1}(1 - z)\frac{B'(z)}{1 - pB(z)} = \frac{1}{b_1}\frac{b_1(1 - pB(z)) - (1 - z)B'(z)}{1 - pB(z)}$$

Denoting the numerator $N(z)$, a sufficient condition is that

$$[z^x] N(z) \geq 0 \text{ for all } x \geq 1$$

But

$$N(z) = \sum_{x \geq 1} z^x [(x - pb_1) b_x - (x+1) b_{x+1}]. \ \square$$

Let us show on four examples that these conditions can be met.

1. Suppose $B(z) = z$. Then $0 = \frac{b_{x+1}}{b_x} \leq \frac{x-p}{x+1}$ for all $x \geq 1$. The simple shifted-geometric rv Δ is SD.

2. Suppose $B(z) = b_1 z + b_2 z^2$ with $b_2 = 1 - b_1$. We need to check conditions under which $\frac{b_2}{b_1} \leq \frac{1-pb_1}{2}$. This condition is met iff the polynomial $pb_1^2 - 3b_1 + 2 \leq 0$ which holds iff $b_1 \geq b_1^*$ where $b_1^* \in (0,1)$ is the zero of this polynomial in $(0,1)$.

3. Suppose $B(z) = \overline{\alpha}z/(1 - \alpha z)$, $\alpha \in (0,1)$, the pgf of a geometric($\overline{\alpha}$) rv, with $b_x = \overline{\alpha}\alpha^{x-1}$. The condition reads: $\alpha \leq \frac{x - p\overline{\alpha}}{x+1}$. It is fulfilled if $\alpha \leq \frac{x-p}{x+q}$ for all $x \geq 1$ which is $\alpha \leq q/(1+q) < 1$ (or $\overline{\alpha} = b_1 \geq b_1^* = 1/(1+q)$).

4. Sibuya. Suppose $B(z) = 1 - (1-z)^\alpha$, $\alpha \in (0,1)$, with $b_x = \alpha [\overline{\alpha}]_{x-1}/x!$, $x \geq 1$ (where $[\overline{\alpha}]_x = \overline{\alpha}(\overline{\alpha}+1)...(\overline{\alpha}+x-1)$, $x \geq 1$ are the rising factorials of $\overline{\alpha}$ and $[\overline{\alpha}]_0 := 1$). The condition reads: $\frac{\overline{\alpha}+x-1}{x+1} \leq \frac{x-p\alpha}{x+1}$ which is always fulfilled. The shifted-geometric rv with Sibuya distributed compounding rv is always SD.

Proposition. *Under the condition that* X_∞ *is SD and so unimodal,* X_∞ *has always mode at the origin.*

Proof: The condition is that, with $\pi(0) = \Phi_\infty(0) = q$ and $\pi(1) = \Phi'_\infty(0) = pqB'(0) = pqb_1$, $\pi(1)/\pi(0) = pb_1 < 1$ which is always satisfied.

2.5. **Green (potential) kernel at** $(0,0)$: **Contact probability at** 0 **and first return time to** 0. Suppose $X_0 = 0$. With then $\Phi_0(z) = 1$, define the double generating function $\Phi(u,z) = \sum_{n\geq 0} u^n \Phi_n(z)$, obeying $\Phi(u,1) = 1/(1-u)$. [Note that $\log z$ was Laplace-conjugate to X_n and now $\log u$ is Laplace-conjugate to n]. Then,

$$\frac{1}{u}\left(\Phi\left(u,z\right)-1\right)=pB\left(z\right)\Phi\left(u,z\right)+q\Phi\left(u,1-c\left(1-z\right)\right)$$

$$\Phi\left(u,z\right)=\frac{1+qu\Phi\left(u,1-c\left(1-z\right)\right)}{1-puB\left(z\right)} \tag{7}$$

$$\Phi\left(u,0\right)=1+qu\Phi\left(u,1-c\right).$$

With $H\left(u,z\right):=1/\left(1-puB\left(z\right)\right)$, upon iterating, with $\Phi\left(0,z\right)=\Phi_0\left(z\right)=1$

$$\Phi\left(u,z\right)=\sum_{n\geq 0}\left(qu\right)^n\prod_{m=0}^{n}H\left(u,1-c^m\left(1-z\right)\right). \tag{8}$$

In particular

$$\Phi\left(u,0\right)=\sum_{n\geq 0}\left(qu\right)^n\prod_{m=0}^{n}H\left(u,1-c^m\right)=1+\sum_{n\geq 1}\prod_{m=1}^{n}\frac{qu}{1-puB\left(1-c^m\right)}. \tag{9}$$

Note that

$$G_{0,0}\left(u\right):=\Phi\left(u,0\right)=\sum_{n\geq 0}u^n\mathbf{P}_0\left(X_n=0\right)=\left(I-uP\right)^{-1}\left(0,0\right)$$

is the Green kernel of the chain at $(0,0)$ (the matrix element $(0,0)$ of the resolvent of P). Consequently,

Proposition. *The Green kernel $G_{0,0}\left(u\right)$ is given by (9).*

With $h_{m'}:=\left[u^{m'}\right]\prod_{m=0}^{n-1}\left(1-puB\left(1-c^m\right)\right)^{-1}$, we have the following expression for the contact probability at 0:

$$\left[u^n\right]\Phi\left(u,0\right)=\Phi_n\left(0\right)=\mathbf{P}_0\left(X_n=0\right)=\sum_{m'=0}^{n-1}q^{n-m'}h_{m'}. \tag{10}$$

Let us have a quick check of this formula. When $n=1$, this leads to $\mathbf{P}_0\left(X_1=0\right)=q$, and, if $n=2$ to $\mathbf{P}_0\left(X_2=0\right)=q^2+q\left[u^1\right]\left(1-puB\left(1-c\right)\right)^{-1}=q^2+pqB\left(1-c\right)$. The second part is not quite trivial because it accounts for any movement up in the first step (wp p) immediately followed by a movement down to 0. This is consistent however with the binomial formula $\mathbf{P}\left(X_1=0\mid X_0=x\right)=q\mathbf{P}\left(c\circ x=0\right)=q\left(1-c\left(1-z\right)\right)^x\mid_{z=0}=q\left(1-c\right)^x$ so that the second part is

$$p\sum_{x\geq 1}\mathbf{P}\left(\beta=x\right)\mathbf{P}\left(X_1=0\mid X_0=x\right)=pq\sum_{x\geq 1}b_x\left(1-c\right)^x=pqB\left(1-c\right).$$

In the positive recurrent case,

$$\mathbf{P}_0\left(X_n=0\right)\to\pi\left(0\right)=\prod_{n\geq 0}\phi_\Delta\left(1-c^n\right)>0.$$

So, in this case,

$$\sum_{m'=0}^{n-1}q^{n-m'}h_{m'}\to\pi\left(0\right)>0 \text{ as } n\to\infty.$$

In the transient case, when n is large, the dominant term in the expression of $\Phi_n(0) = \mathbf{P}_0(X_n = 0)$ is when $m' = n-1$, which is $q(pB(1-c))^n$. For large n, $\mathbf{P}_0(X_n = 0)$ decays geometrically with n at rate $pB(1-c)$.

We finally observe that an expression of

$$h_{m'} = \left[u^{m'}\right] \prod_{m=0}^{n-1} (1 - puB(1-c^m))^{-1}$$

can be obtained from a decomposition into simple elements of the product.

The Green kernel at $(0,0)$ is thus $G_{0,0}(u) = \Phi(u,0)$.

If $n \geq 1$, from the recurrence $\mathbf{P}_0(X_n = 0) = P^n(0,0) = \sum_{m=0}^{n} \mathbf{P}(\tau_{0,0} = m) P^{n-m}(0,0)$, we see that the pgf $\phi_{0,0}(u) = \mathbf{E}(u^{\tau_{0,0}})$ of the first return time to 0, $\tau_{0,0}$ and $G_{0,0}(u)$ are related by the Feller relation (see [3] pp $3-4$ for example)

$$G_{0,0}(u) = \frac{1}{1-\phi_{0,0}(u)} \text{ and } \phi_{0,0}(u) = \frac{G_{0,0}(u)-1}{G_{0,0}(u)}.$$

Hence,

Proposition. *The pgf $\phi_{0,0}(u)$ of the first return time $\tau_{0,0}$ is*

$$\phi_{0,0}(u) = 1 - \frac{1}{G_{0,0}(u)}, \tag{11}$$

where $G_{0,0}(u)$ is given by (9).

Note

$$G_{0,0}(1) = \sum_{n\geq 0} \mathbf{P}_0(X_n = 0) = 1 + \sum_{n\geq 1} q^n \prod_{m=0}^{n} H(1, 1-c^m) = \infty$$

if and only if X is recurrent, [20], [22]. And in that case, $\phi_{0,0}(1) = \mathbf{P}(\tau_{0,0} < \infty) = 1 - \frac{1}{G_{0,0}(1)} = 1$. Positive-(- null) recurrence is when $\phi'_{0,0}(1) = \mathbf{E}(\tau_{0,0}) = 1/\pi_0 < \infty$ $(= \infty)$. Note finally $G_{0,0}(0) = 1$ so that $\phi_{0,0}(0) = \mathbf{P}\left(\tau_{0,0} = 0\right) = 0$.

2.6. **Starting from $x_0 > 0$: Green kernel at $(x_0, 0)$ and first extinction time $\tau_{x_0,0}$.** Suppose now $X_0 = x_0 > 0$. After shifting X_n of x_0, with $\Phi(u,z) = \sum_{n\geq 0} u^n \mathbf{E}\left(z^{x_0+X_n}\right)$, we now get

$$\frac{1}{u}\left(\Phi(u,z) - z^{x_0}\right) = pB(z)\Phi(u,z) + q\Phi(u, 1-c(1-z))$$

$$\Phi(u,z) = \frac{z^{x_0} + qu\Phi(u, 1-c(1-z))}{1-puB(z)} \tag{12}$$

$$\begin{aligned} \Phi(u,z) &= \sum_{n\geq 0} (qu)^n (1-c^n(1-z))^{x_0} \prod_{m=0}^{n} H(u, 1-c^m(1-z)) \\ \Phi(u,0) &= \sum_{n\geq 0} (qu)^n (1-c^n)^{x_0} \prod_{m=0}^{n} H(u, 1-c^m). \end{aligned}$$

We obtained:

Proposition. *The contact probability at* 0 *for the chain started at* $x_0 > 0$ *is given by*

$$[u^n]\,\Phi(u,0) = \Phi_n(0) = \mathbf{P}_{x_0}(X_n = 0) = \sum_{m'=0}^{n-1}\left(1 - c^{n-m'}\right)^{x_0} q^{n-m'} h_{m'}. \tag{13}$$

Let us give the first two terms. As required, when $n = 1$, $\mathbf{P}_{x_0}(X_1 = 0) = q(1-c)^{x_0}$ and when $n = 2$,

$$\mathbf{P}_{x_0}(X_2 = 0) = q^2\left(1 - c^2\right)^{x_0} + pq(1-c)^{x_0} B(1-c)$$

a weighted sum of the two terms appearing in the expression of $\mathbf{P}_0(X_2 = 0)$.

Corrollary.

(i) *When* x_0 *is large and* n *fixed, the small but dominant term is when* $m' = 0$ *which is* $q^n(1-c^n)^{x_0}$. *So* $\mathbf{P}_{x_0}(X_n = 0)$ *decays geometrically with* x_0. *This expression quantifies the probability that the population is in an early state of extinction given the initial population size was large. Early is when* $c^n \gg 1/x_0$ *(so that* $(1-c^n)^{x_0} \ll (1-1/x_0)^{x_0} \ll e^{-1}$*), so when* $n \ll -\log_c x_0$.

(ii) *In the transient case, when* n *is large and* x_0 *is fixed, the dominant term is when* $m' = n-1$ *which is* $q(1-c)^{x_0}(pB(1-c))^n$. *So* $\mathbf{P}_{x_0}(X_n = 0)$ *decays geometrically with* n *at rate* $pB(1-c)$. *In the positive recurrent case,* $\mathbf{P}_{x_0}(X_n = 0) \to \pi(0) > 0$ *as* $n \to \infty$*, independently of* x_0.

• **The Green kernel at** $(x_0, 0)$ is thus $G_{x_0,0}(u) = \left[z^0\right]\Phi(u,z)$. It is related to the pgf of the first extinction time $\tau_{x_0,0}$ by the Feller relation

$$\phi_{x_0,0}(u) = \mathbf{E}\left(u^{\tau_{x_0,0}}\right) = \frac{G_{x_0,0}(u)}{G_{0,0}(u)}.$$

Therefore,

Proposition. *With* $x_0 > 0$*, the pgf of the first extinction time* $\tau_{x_0,0}$ *is*

$$\phi_{x_0,0}(u) = \mathbf{E}\left(u^{\tau_{x_0,0}}\right) = \frac{\sum_{n\geq 1}(qu)^n(1-c^n)^{x_0}\prod_{m=0}^{n} H(u, 1-c^m)}{\sum_{n\geq 0}(qu)^n\prod_{m=0}^{n} H(u, 1-c^m)}. \tag{14}$$

In the recurrent case, state 0 is visited infinitely often and so both $G_{0,0}(1)$ and $G_{x_0,0}(1) = \infty$, and

$$\mathbf{P}(\tau_{x_0,0} < \infty) = \phi_{x_0,0}(1) = \frac{G_{x_0,0}(1)}{G_{0,0}(1)} = 1.$$

In the sequel, we shall let $\overline{P}$ stand for the substochastic transition matrix obtained from P while deleting its first row and column.

• An alternative representation of $\phi_{x_0,0}(u)$ follows from the following classical first-step analysis:

Let $X_1(x_0)$ be the position of X_n started at x_0.

Let $X_+(x_0)$ be a positive rv with $\mathbf{P}(X_+(x_0) = y) = \overline{P}(x_0, y)/\sum_{y\geq 1}\overline{P}(x_0, y)$, $y \geq 1$. With $\tau'_{X_+(x_0),0}$ a statistical copy of $\tau_{X_+(x_0),0}$, first-step analysis yields:

$$\tau_{x_0,0} \overset{d}{=} 1 \cdot \mathbf{1}_{\{X_1(x_0)=0\}} + \mathbf{1}_{\{X_1(x_0)>0\}} \cdot \left(1 + \tau'_{X_+(x_0),0}\right).$$

Clearly, $\mathbf{P}\left(X_1\left(x_0\right)=0\right)=P\left(x_0,0\right)=q\left(1-c\right)^{x_0}=:q_{x_0}$, $\mathbf{P}\left(X_1\left(x_0\right)>0\right)=:p_{x_0}=\sum_{y\geq 1}\overline{P}\left(x_0,y\right)=1-q_{x_0}$. Therefore $\phi_{x_0,0}\left(u\right):=\mathbf{E}\left(u^{\tau_{x_0,0}}\right)$, obeys the recurrence $\phi_{x_0,0}\left(u\right)=q_{x_0}u+up_{x_0}\mathbf{E}\phi_{X_+(x_0),0}\left(u\right)$.

With $\boldsymbol{\phi}\left(u\right)=\left(\phi_{1,0}\left(u\right),\phi_{2,0}\left(u\right),...\right)'$ the column-vector of the $\phi_{x_0,0}\left(u\right)=\mathbf{E}u^{\tau_{x_0,0}}$, and $\mathbf{q}=\left(q_1,q_2,...\right)'$ the first column-vector of the matrix P, $\boldsymbol{\phi}\left(u\right)$ then solves:

$$\boldsymbol{\phi}\left(u\right)=u\mathbf{q}+u\overline{P}\boldsymbol{\phi}\left(u\right), \tag{15}$$

whose formal solution is (compare with the explicit expression (14))

$$\boldsymbol{\phi}\left(u\right)=u\left(I-u\overline{P}\right)^{-1}\mathbf{q}=:u\overline{G}\left(u\right)\mathbf{q},$$

involving the resolvent matrix $\overline{G}\left(u\right)$ of $\overline{P}$. Note $\boldsymbol{\phi}\left(1\right)=\left(I-\overline{P}\right)^{-1}\mathbf{q}$ gives the column-vector of the probabilities of eventual extinction $\boldsymbol{\phi}\left(1\right):=\left(\phi_{1,0}\left(1\right),\phi_{2,0}\left(1\right),...\right)'$, so with $\phi_{x_0,0}\left(1\right)=\mathbf{P}\left(\tau_{x_0,0}<\infty\right)$. Clearly $\boldsymbol{\phi}\left(1\right)=\mathbf{1}$ (the all-one column vector) in the recurrent case. In that case, from (15), introducing the column vector $\mathbf{E}\left(\tau_{.,0}\right):=\left(\mathbf{E}\left(\tau_{1,0}\right),\mathbf{E}\left(\tau_{2,0}\right),...\right)'$ where $\mathbf{E}\left(\tau_{x_0,0}\right)=\phi_{x_0,0}'\left(1\right)$ and observing $\mathbf{q}+\overline{P}\boldsymbol{\phi}\left(1\right)=\mathbf{1}$, we get

$$\boldsymbol{\phi}'\left(1\right):=\mathbf{E}\left(\tau_{.,0}\right)=\mathbf{1}+\overline{P}\mathbf{E}\left(\tau_{.,0}\right),\text{ equivalently }\mathbf{E}\left(\tau_{.,0}\right)=\left(I-\overline{P}\right)^{-1}\mathbf{1}=\overline{G}\mathbf{1},\text{ or}$$

$$\mathbf{E}\left(\tau_{x_0,0}\right)=\sum_{y\geq 1}\overline{G}_{x_0,y}.$$

• Yet an alternative representation $\boldsymbol{\phi}\left(u\right):=\left(\phi_{1,0}\left(u\right),\phi_{2,0}\left(u\right),...\right)'$ is as follows. From the identity

$$\overline{P}^n\left(x_0,y\right)=\mathbf{P}_{x_0}\left(X_n=y,\tau_{x_0,0}>n\right),$$

we get $\mathbf{P}\left(\tau_{.,0}>n\right)=\overline{P}^n\mathbf{1}$,as a column vector, and so,

$$\sum_{n\geq 0}u^n\mathbf{P}\left(\tau_{.,0}>n\right)=\overline{G}\left(u\right)\mathbf{1}.$$

This leads in particular, as expected, to $\mathbf{E}\left(\tau_{.,0}\right)=\overline{G}\left(1\right)\mathbf{1}$ and to (compare with the explicit expression (14))

$$\boldsymbol{\phi}\left(u\right)=\sum_{n\geq 0}u^n\mathbf{P}\left(\tau_{.,0}=n\right)=\mathbf{1}-\left(1-u\right)\overline{G}\left(u\right)\mathbf{1}.$$

We obtained:

Proposition: *We have* $\boldsymbol{\phi}\left(1\right)=\mathbf{P}\left(\tau_{.,0}<\infty\right)$ *and so* $\overline{G}\left(1\right)\mathbf{1}<\infty\Rightarrow\mathbf{P}\left(\tau_{.,0}<\infty\right)=\mathbf{1}$*, meaning recurrence of* X_n*. In fact, positive recurrence is precisely when* $\mathbf{E}\left(\tau_{.,0}\right)=\overline{G}\left(1\right)\mathbf{1}<\infty$*. If* $\overline{G}\left(1\right)\mathbf{1}=\infty$*, the chain is null-recurrent if* $\left(1-u\right)\overline{G}\left(u\right)\mathbf{1}\to\mathbf{0}$ as $u\to 1$*, transient if* $\left(1-u\right)\overline{G}\left(u\right)\mathbf{1}\to\mathbf{P}\left(\tau_{.,0}=\infty\right)$ *as* $u\to 1$*, a non-zero limit.*

The matrix $\overline{P}$ is substochatic with spectral radius $\rho\in\left(0,1\right)$. With $\mathbf{r}$ and $\mathbf{l}'$ the corresponding right and left positive eigenvectors of $\overline{P}$, so with $\overline{P}\mathbf{r}=\rho\mathbf{r}$ and $\mathbf{l}'\overline{P}=\rho\mathbf{l}'$, $\overline{P}^n\sim\rho^n\cdot\mathbf{r}\mathbf{l}'$ (as n is large) where $\mathbf{r}\mathbf{l}'$ is the projector onto the first eigenspace. By Perron-Frobenius theorem, [27], [16], if we normalize $\mathbf{l}$ to be of l_1- norm one, we get

Proposition: *In the positive recurrent case for* X_n ($\mathbf{E}\log_+\Delta<\infty$):

(i) With $r(x_0)$ the x_0-entry of $\mathbf{r}$,

$$\rho^{-n}\mathbf{P}(\tau_{x_0,0}>n)\to r(x_0),$$

showing that $\mathbf{P}(\tau_{x_0,0}>n)$ has geometric tails with rate ρ (fast extinction time).

(ii) With $l(y)$ the $y-$entry of $\mathbf{l}$, for all $x_0>0$,

$$\mathbf{P}_{x_0}(X_n=y\mid\tau_{x_0,0}>n)\to l(y),\ \text{as}\ n\to\infty,$$

showing that the left eigenvector $\mathbf{l}$ is the quasi-stationary distribution (or Yaglom limit, [29]).

Proof: In this case, with $R=\rho^{-1}>1$, the convergence radius of $\overline{G}$, $\overline{G}(R)=\infty$ and $\overline{P}$ is $R-$positive recurrent. (i) follows from $\mathbf{P}(\tau_{\cdot,0}>n)=\overline{P}^n\mathbf{1}$ and (ii) from $\mathbf{P}_{x_0}(X_n=y\mid\tau_{x_0,0}>n)=\overline{P}^n(x_0,y)/\overline{P}^n\mathbf{1}$.

Remark. The full Green kernel at (x_0,y_0) is $G_{x_0,y_0}(u)=[z^{y_0}]\,\Phi(u,z)$. Hence

$$G_{x_0,y_0}(u)=\sum_{n\geq0}(qu)^n[z^{y_0}](1-c^n(1-z))^{x_0}\prod_{m=0}^{n}H(u,1-c^m(1-z)) \tag{16}$$

$$=\sum_{n\geq0}(qu)^n\sum_{y=0}^{y_0}h_{n,y}(u)\,g_{n,y_0-y}$$

where

$$g_{n,y}=[z^y](1-c^n(1-z))^{x_0}=\binom{x_0}{y}c^{ny}(1-c^n)^{x_0-y}$$

and

$$h_{n,y}(u)=[z^y]\prod_{m=0}^{n}H(u,1-c^m(1-z)),$$

which can be obtained from a decomposition into simple elements of the inner product.

Using $P^n(x_0,y_0)=\sum_{m=1}^{n}\mathbb{P}(\tau_{x_0,y_0}=m)\,P^{n-m}(y_0,y_0)$, $n\geq1$, we easily get the expression of the pgf of the first hitting times $\tau_{x_0,y_0}=\inf(n\geq1:X_n=y_0\mid X_0=x_0)$, as

$$\phi_{x_0,y_0}(z)=\frac{G_{x_0,y_0}(z)}{G_{y_0,y_0}(z)}. \tag{17}$$

2.7. **Average position of X_n started at x_0.** The double-generating function $\Phi(u,z)$ can be used to compute $\mathbf{E}_{x_0}(X_n)$. With $\Phi'(u,z)=\partial_z\Phi(u,z)$

$$\begin{aligned}
\Phi'(u,1) &= x_0\sum_{n\geq0}(qcu)^n H(u,1)^{n+1}+\frac{p\rho}{1-(p+qc)}\left(\frac{1}{1-u}-\frac{1}{1-u(p+qc)}\right)\\
&= x_0\frac{H(u,1)}{1-qcuH(u,1)}+\frac{p\rho}{1-(p+qc)}\left(\frac{1}{1-u}-\frac{1}{1-u(p+qc)}\right)\\
&= \frac{x_0}{1-(p+qc)\,u}+\frac{p\rho}{1-(p+qc)}\left(\frac{1}{1-u}-\frac{1}{1-u(p+qc)}\right)
\end{aligned}$$

This shows that, in the positive recurrent case, with $\mu_n = \mathbf{E}_{x_0}(X_n)$,

$$\mu_n = [u^n]\,\Phi'(u,1) = x_0 (p+qc)^n + \frac{p\rho}{q(1-c)}\left(1-(1-q(1-c))^n\right) \tag{18}$$

$$= x_0 (p+qc)^n + \mathbf{E}_0(X_n) \to \frac{p\rho}{q(1-c)} \text{ whatever } x_0,$$

solving

$$\mu_{n+1} = (p+qc)\,\mu_n + p\rho,\ \mu_0 = x_0.$$

2.8. **The height** H **of an excursion.** Assume $X_0 = x_0$ and consider a version of X_n which is absorbed at 0. Let $X_{n\wedge\tau_{x_0,0}}$ stopping X_n when it first hits 0. Let us define the scale (or harmonic) function φ of X_n as the function which makes $Y_n \equiv \varphi\left(X_{n\wedge\tau_{x_0,0}}\right)$ a martingale. The function φ is important because, for all $0 \leq x_0 < h$, with $\tau_{x_0} = \tau_{x_0,0} \wedge \tau_{x_0,h}$ the first hitting time of $\{0,h\}$ starting from x_0 (assuming $\varphi(0) \equiv 0$)

$$\mathbf{P}\left(X_{\tau_{x_0}} = h\right) = \mathbf{P}\left(\tau_{x_0,h} < \tau_{x_0,0}\right) = \frac{\varphi(x_0)}{\varphi(h)},$$

resulting from

$$\mathbf{E}\varphi\left(X_{n\wedge\tau_{x_0}}\right) = \varphi(x_0) = \varphi(h)\,\mathbf{P}\left(\tau_{x_0,h} < \tau_{x_0,0}\right) + \varphi(0)\,\mathbf{P}\left(\tau_{x_0,h} > \tau_{x_0,0}\right).$$

• **The case** $\beta_1 \sim \delta_1$**.** Let us consider the height H of an excursion of the original MC X_n first assuming $\beta_1 \sim \delta_1$ (a birth event adds only one individual). With probability q, $H = 0$ and with probability p, starting from $X_1 = 1$, it is the height of a path from state 1 to 0 of the absorbed process X_n. Using this remark, the event $H = h$ is realized when $\tau_{1,h} < \tau_{1,0}$ and $\tau_{h,h+1} > \tau_{h,0}$, the latter two events being independent. Thus (with $\mathbf{P}(H=0) = q$):

$$\mathbf{P}(H=h) = p\frac{\varphi(1)}{\varphi(h)}\left(1 - \frac{\varphi(h)}{\varphi(h+1)}\right),\ h \geq 1. \tag{19}$$

We clearly have $\sum_{h\geq 1}\mathbf{P}(H=h) = p$ because partial sums form a telescoping series. But (19) is also

$$\mathbf{P}(H \geq h) = 1/\varphi(h),\ h \geq 1, \tag{20}$$

with $\varphi(1) = 1/p$.

It remains to compute φ with $\varphi(0) = 0$ and $\mathbf{P}(H \geq 1) = 1/\varphi(1) = p$. We wish to have: $\mathbf{E}_{x_0}(Y_{n+1} \mid Y_n = x) = x$, leading to

$$\varphi(x) = p\varphi(x+1) + q\sum_{y=1}^{x}\binom{x}{y}c^y(1-c)^{x-y}\varphi(y),\ x_0 \geq 1.$$

The vector $\boldsymbol{\varphi}$ is the right eigenvector associated to the eigenvalue 1 of the modified version P^* of the stochastic matrix P having 0 as an absorbing state: ($P^*(0,0) = 1$), so with: $\boldsymbol{\varphi} = P^*\boldsymbol{\varphi}$, $\varphi(0) = 0$, [21]. The searched 'harmonic' function is increasing and given by recurrence, $\varphi(1) = 1/p$ and

$$\varphi(x+1) = \frac{1}{p}\left(\varphi(x)\left[1-qc^x\right] - q\sum_{y=1}^{x-1}\binom{x}{y}c^y(1-c)^{x-y}\varphi(y)\right),\ x \geq 1 \tag{21}$$

The first two terms are

$$\begin{aligned}\varphi(2) &= \frac{1}{p}(1-qc)\varphi(1) = \frac{1}{p^2}(1-qc)\\ \varphi(3) &= \frac{1}{p}\left(\varphi(2)\left[1-qc^2\right] - 2qc(1-c)\varphi(1)\right)\\ &= \frac{1}{p^3}(1-qc)\left(1-qc^2\right) - 2\frac{q}{p}c(1-c).\end{aligned}$$

The sequence $\varphi(x)$ is diverging when the chain X is recurrent.

Proposition. *When $\beta \sim \delta_1$, equations (20) and (21) characterize the law of the excursion height H of the random walker in the recurrent case. In the transient case, $\varphi(x)$ converges to a value φ^* and $\mathbf{P}(H=\infty) = 1/\varphi^* = \mathbf{P}(\tau_{0,0}=\infty)$.*

• **General β_1.** Whenever the law of β is general, the matrix P^* is no longer lower Hessenberg and the harmonic vector $\boldsymbol{\varphi} = P^*\boldsymbol{\varphi}$, with $\varphi(0)=0$, cannot be obtained by a recurrence. However, the event $H \geq h \geq 1$ is realized whenever a first birth event occurs with size $\beta_1 \geq h$ or, if $\beta_1 < h$, whenever for all states $h' \geq h$ being hit when the amplitude β of a last upper jump is larger than $h'-h$, then $\tau_{\beta_1,h'} < \tau_{\beta_1,0}$. Hence,

Proposition. *For a recurrent walker with general β, $\mathbf{P}(H=\infty)=0$ where, when $h \geq 1$,*

$$\begin{aligned}\mathbf{P}(H \geq h) &= p\mathbf{P}(\beta_1 \geq h) + p\sum_{x=1}^{h-1}\mathbf{P}(\beta_1 = x)\sum_{h' \geq h}\frac{\varphi(x)}{\varphi(h')}\mathbf{P}(\beta > h'-h)\\ &= p\mathbf{P}(\beta_1 \geq h) + p\cdot\sum_{x=1}^{h-1}\mathbf{P}(\beta_1 = x)\varphi(x)\cdot\sum_{h'' \geq 0}\frac{1}{\varphi(h+h'')}\mathbf{P}(\beta > h'')\end{aligned}$$

generalizing (20).

The harmonic function is also important in the construction of the absorbed process conditioned on being currently alive and its quasi-stationary limit law, [29], (see below).

2.9. **Estimation from an N-sample of X, say $(x_0, x_1, ..., x_N)$.** From the transition matrix $P(x,y)$, the log-likelihood function of the N-sample is

$$L(x_0, x_1, ..., x_N) = \sum_{n=1}^{N}\left[\log\left(qd_{x_{n-1},x_n}\right)\mathbf{1}(x_n \leq x_{n-1}) + \log\left(pb_{x_n - x_{n-1}}\right)\log\mathbf{1}(x_n > x_{n-1})\right]$$

If one knows that some population grows and decays according to the binomial catastrophe model with $\mathbf{E}\beta = \rho < \infty$, we propose the following estimators: the MLE estimator of p while setting $\partial_p L = 0$ is

$$\widehat{p} = \frac{1}{N}\sum_{n=1}^{N}\mathbf{1}(x_n > x_{n-1}).$$

With $\rho = \mathbf{E}\beta < \infty$, if the law of β is a known one-parameter $\rho-$family of pmfs,

$$\widehat{\rho} = \frac{1}{\sum_{n=1}^{N} \mathbf{1}(x_n > x_{n-1})} \sum_{n=1}^{N} (x_n - x_{n-1}) \mathbf{1}(x_n > x_{n-1})$$

Note that if

$$\widehat{\rho p} = \frac{1}{N} \sum_{n=1}^{N} (x_n - x_{n-1}) \mathbf{1}(x_n > x_{n-1})$$

then $\widehat{\rho p} = \widehat{\rho}\widehat{p}$.

Also, in view of $\mathbf{P}(X_{n+1} = x \mid X_n = x) = qc^x$, $x \geq 0$,

$$\widehat{c} = \frac{1}{\sum_{n=1}^{N} \mathbf{1}(x_n = 1)} \sum_{n=1}^{N} \mathbf{1}(x_n = 1, x_{n-1} = 1)$$

2.10. **Absorbed version of X_n.** We consider again a version of X_n started at $x_0 > 0$, but which is now absorbed when first hitting 0. We work under the condition that the original (non-absorbed) process is not transient, because if it were, so would its absorbed version with a positive probability to drift to ∞. The only thing which changes in the transition matrix P as from (1) is its first row which becomes $P(0, y) = \delta_{0,y}$, $y \geq 0$. Letting $\Phi_n(z) = \mathbf{E}\left(z^{X_n}\right)$ with $\Phi_0(z) = z^{x_0}$, taking into account the behavior of X_n at 0, with $\Phi_0(z) = z^{x_0}$,the recurrence (2) now becomes

$$\Phi_{n+1}(z) = \Phi_n(0) + pB(z)(\Phi_n(z) - \Phi_n(0)) + q[\Phi_n(1 - c(1 - z)) - \Phi_n(0)]$$

$$\Phi_{n+1}(z) = p(1 - B(z))\Phi_n(0) + pB(z)\Phi_n(z) + q\Phi_n(1 - c(1 - z))$$

$$\Phi_{n+1}(z) - \Phi_n(z) = p(B(z) - 1)(\Phi_n(z) - \Phi_n(0)) + q(\Phi_n(1 - c(1 - z)) - \Phi_n(z))$$

One can check that for all $z \in [0, 1)$,

$$\Phi_\infty(z) = 1, \text{ consistently with } X_\infty = 0.$$

Note also

$$\begin{aligned}
\Phi_{n+1}(1) &= \Phi_n(1) = \Phi_0(1) = 1 \text{ (no mass loss)} \\
\Phi_{n+1}(0) &= p\Phi_n(0) + q\Phi_n(1 - c) \text{ else} \\
\Phi_{n+1}(0) - \Phi_n(0) &= q(\Phi_n(1 - c) - \Phi_n(0)) > 0 \text{ if } n \geq 1,
\end{aligned}$$

showing that $\Phi_n(0) = \mathbf{P}(X_n = 0 \mid X_0 = x_0)$ is increasing tending to 1 under the recurrence condition. Finally,

- $\Phi_n(0) = 1 \Leftrightarrow X_n = 0 \Leftrightarrow \Phi_n(z) = 1 \Rightarrow \Phi_n(1 - c) = 1 \Rightarrow \Phi_{n+1}(0) = 1$ (0 is absorbing)

- $X_n = 0 \Leftrightarrow \tau_{x_0,0} \leq n \Rightarrow \Phi_n(0) = \mathbf{P}(\tau_{x_0,0} \leq n)$.

- $\mathbf{P}(\tau_{x_0,0} = n) = \mathbf{P}(\tau_{x_0,0} \leq n) - \mathbf{P}(\tau_{x_0,0} \leq n - 1) = \Phi_n(0) - \Phi_{n-1}(0) > 0$.

Here $\tau_{x_0,0}$ be the first (and last) hitting time of 0 for X_n given $X_0 = x_0 > 0$. It has the same distribution as the one obtained for the original non-absorbed chain. Considering the transition matrix of X_n where the first row and column have been removed, the dynamics of

$$\Psi_n(z) = \mathbf{E}\left(z^{X_n} \mathbf{1}_{\tau_{x_0,0} > n}\right)$$

is

$$\Psi_{n+1}(z) = pB(z)\Psi_n(z) + q\left[\Psi_n(1-c(1-z)) - \Psi_n(1-c)\right], \; \Psi_0(z) = z^{x_0}, \; \Psi_0(0) = 0$$

Note $\Psi_n(0) = 0$ entails $\Psi_{n+1}(0) = 0$: as required, there is no probability mass at 0.

Furthermore, $\Psi_{n+1}(1) = p\Psi_n(1) + q\left[\Psi_n(1) - \Psi_n(1-c)\right] = \Psi_n(1) - q\Psi_n(1-c)$, so that $\Psi_{n+1}(1) - \Psi_n(1) < 0$ translating a natural loss of mass.

We have $\Psi_n(1) = \mathbf{P}(\tau_{x_0,0} > n)$ (note $\Psi_n(1) = 1 - \Phi_n(0)$). Conditioning, with $\Phi_n^c(z) = \mathbf{E}\left(z^{X_n} \mid \tau_{x_0,0} > n\right)$, upon normalizing, we get

Proposition. *Under the positive recurrence condition for X_n, as $n \to \infty$*

$$\Phi_n^c(z) := \frac{\Psi_n(z)}{\Psi_n(1)} \to \frac{\Psi_\infty(z)}{\Psi_\infty(1)},$$

the pgf of a quasi-stationary distribution, [29]*. An expression of $\Psi_n(z)$ follows from (22), (23) below.*

The double generating function of $\Psi_n(z)$ is

$$\frac{1}{u}\left(\Psi(u,z) - z^{x_0}\right) = pB(z)\Psi(u,z) + q\Psi(u,1-c(1-z)) - q\Psi(u,1-c)$$

Its iterated version is

$$\Psi(u,z) = \sum_{n\geq 0}(qu)^n\left(1-c^n(1-z)\right)^{x_0}\prod_{m=0}^{n}H\left(u,1-c^m(1-z)\right) \tag{22}$$

$$-\Psi(u,1-c)\sum_{n\geq 1}(qu)^n\prod_{m=0}^{n-1}H\left(u,1-c^m(1-z)\right).$$

If $z = 1-c$,

$$\begin{aligned}\Psi(u,1-c) &= \sum_{n\geq 0}(qu)^n\left(1-c^{n+1}\right)^{x_0}\prod_{m=0}^{n}H\left(u,1-c^{m+1}\right)\\ &\quad -\Psi(u,1-c)\sum_{n\geq 1}(qu)^n\prod_{m=0}^{n-1}H\left(u,1-c^{m+1}\right),\end{aligned}$$

so that

$$\Psi(u,1-c) = \frac{\sum_{n\geq 0}(qu)^n\left(1-c^{n+1}\right)^{x_0}\prod_{m=0}^{n}H\left(u,1-c^{m+1}\right)}{1+\sum_{n\geq 1}(qu)^n\prod_{m=1}^{n}H\left(u,1-c^m\right)}. \tag{23}$$

Plugging (23) into (22) yields a closed form expression of $\Psi(u,z)$ and then of

$$\frac{\Psi_n(z)}{\Psi_n(1)} = \frac{[u^n]\Psi(u,z)}{[u^n]\Psi(u,1)}.$$

The value of $\Psi(u,z)$ at $z=1$ is $\Psi(u,1)=\sum_{n\geq 0}u^n\mathbf{P}(\tau_{x_0,0}>n)$ so that $\Psi(u,1)=\left(1-\mathbf{E}u^{\tau_{x_0,0}}\right)/(1-u)$. With $H(u,1)=1/(1-pu)$, from (22)

$$\begin{aligned}
\Psi(u,1) &= \sum_{n\geq 0}(qu)^n\prod_{m=0}^{n}H(u,1)-\Psi(u,1-c)\sum_{n\geq 1}(qu)^n\prod_{m=0}^{n-1}H(u,1)\\
&= \frac{1}{1-pu}\frac{1}{1-qu/(1-pu)}-\Psi(u,1-c)\frac{qu}{1-pu}\frac{1}{1-qu/(1-pu)}\\
&= \frac{1}{1-u}-\Psi(u,1-c)\frac{qu}{1-u}=\frac{1}{1-u}\left(1-qu\Psi(u,1-c)\right).
\end{aligned}$$

The pgf of $\tau_{x_0,0}$ is thus

$$\mathbf{E}u^{\tau_{x_0,0}}=qu\Psi(u,1-c),$$

with $\Psi(u,1-c)$ given by (23). This is consistent with (14).

3. A variant of the binomial catastrophe model

Suppose we are interested in the following simple semi-stochastic decay/surge model: at each step of its evolution, the size of some population either grows by a random number of individuals (as a result of immigration) or shrinks by only one unit (as a result of aging and death of the inlanders). The above binomial catastrophe model is not able to represent this scenario where at least one individual is removed from the population at catastrophic events. To remedy this, we therefore define and study a variant of the above binomial model whereby the transition probabilities in the bulk and at 0 are slightly modified in order to account for the latter decay/surge situation.

- If $X_n\geq 1$, define

$$\begin{aligned}
X_{n+1} &= 1\circ X_n+\beta_{n+1}=X_n+\beta_{n+1}\text{ wp }p\\
&= c\circ(X_n-1)\text{ wp }q
\end{aligned}$$

Given a move down wp q: **One** individual out of X_n is **systematically** removed from the population ($X_n\to X_n-1$); each individual among the X_n-1 remaining ones being independently subject to survival/death (wp c / $1-c$) in the next generation.

- If $X_n=0$ ($p_0+q_0=1$)

$$\begin{aligned}
X_{n+1} &= \beta_{n+1}\text{ wp }p_0\\
&= 0\text{ wp }q_0
\end{aligned}$$

Unless $p_0=p$, our model yields some additional control on the future of the population once it hits 0 (extinction event).

The one-step-transition matrix P of MC X_n is: with $b_x:=\mathbf{P}(\beta=x)$, $x\geq 1$

$$\begin{aligned}
P(0,0) &= q_0,\ P(0,y)=p_0\delta_1,\ y\geq 1\\
P(x,x) &= 0\text{ if }x\geq 1\\
P(x,y) &= q\binom{x-1}{y}c^y(1-c)^{x-1-y},\ x\geq 1\text{ and }0\leq y<x\\
P(x,y) &= p\mathbf{P}(\beta=y-x)=pb_{y-x},\ x\geq 1\text{ and }y>x
\end{aligned}$$

Note that because at least one individual dies out in a shrinking event, the diagonal terms $P(x,x)$ of P are now 0 for all $x \geq 1$.

Remark. *One can introduce a holding probability r_x to stay in state x given $X_n = x$, filling up now the diagonal of P. This corresponds to a time change while considering a modified transition matrix $\widetilde{P}$ where: $p \to p_x = p(1-r_x)$ and $q \to q_x = q(1-r_x)$ $(p_x + q_x + r_x = 1)$. So, with $\boldsymbol{\rho} := \mathbf{1} - \mathbf{r}$, a column vector with entries $\rho_x = 1 - r_x$,*

$$P \to \widetilde{P} = I + D_{\boldsymbol{\rho}}(P - I).$$

If the invariant measure $\boldsymbol{\pi}$ obeying $\boldsymbol{\pi}' = \boldsymbol{\pi}' P$, (the fixed point of $\boldsymbol{\pi}'_{n+1} = \boldsymbol{\pi}'_n P$) exists, then the one of $\widetilde{P}$ also exists and obeys: $\widetilde{\boldsymbol{\pi}}' D_{\boldsymbol{\rho}} = \boldsymbol{\pi}'$.

Suppose first $X_0 = 0$. Let $\mathbf{E} z^{X_n} := \Phi_n(z) = \overline{\Phi}_n(z) + \Phi_n(0)$ with $\Phi_0(z) = 1$ translating $X_0 = 0$. Then, with $p + q = 1$,

$$\begin{aligned}
\Phi_{n+1}(z) &= (q_0 + p_0 z)\Phi_n(0) + pB(z)\overline{\Phi}_n(z) + \frac{q}{1-c(1-z)}\overline{\Phi}_n(1-c(1-z)), \ \overline{\Phi}_0(z) = 0 \\
\Phi_{n+1}(0) &= q_0 \Phi_n(0) + \frac{q}{1-c}\overline{\Phi}_n(1-c), \ \Phi_0(0) = 1.
\end{aligned}$$

Note $\Phi_n(0) = \mathbf{P}(X_n = 0)$ and $\overline{\Phi}_n(0) = 0$ for each $n \geq 0$.

Thus, if these fixed point quantities exist

$$\begin{aligned}
\overline{\Phi}_\infty(z) &= p_0(z-1)\Phi_\infty(0) + pB(z)\overline{\Phi}_\infty(z) + \frac{q}{1-c(1-z)}\overline{\Phi}_\infty(1-c(1-z)) \\
\Phi_\infty(0) &= \frac{q}{p_0(1-c)}\overline{\Phi}_\infty(1-c).
\end{aligned}$$

We shall iterate the first fixed point equation which makes sense only when $c \neq 0$.

With $C(z) = \frac{p_0(z-1)}{1-pB(z)}$ and $D(z) = \frac{q}{1-c(1-z)}\frac{1}{1-pB(z)}$, we get

$$\begin{aligned}
\overline{\Phi}_\infty(z) &= \Phi_\infty(0)\sum_{n\geq 0} C(1-c^n(1-z))\prod_{m=0}^{n-1} D(1-c^m(1-z)) \\
&= \Phi_\infty(0)\left[C(z) + D(z)\sum_{n\geq 1} C(1-c^n(1-z))\prod_{m=1}^{n-1} D(1-c^m(1-z))\right] \\
&= \frac{p_0\Phi_\infty(0)(z-1)}{1-pB(z)}\left[1 + \sum_{n\geq 1}(cq)^n \prod_{m=1}^{n}\frac{1}{1-c^m(1-z)}\frac{1}{1-pB(1-c^m(1-z))}\right]
\end{aligned}$$

Except when $c = 1$, the term inside the bracket has no pole at $z = 1$. Then $\overline{\Phi}_\infty(1) = 0$ and so, assuming $\overline{\Phi}_\infty(0) = 0$, $\overline{\Phi}_\infty(z) = 1$ for all $z \in [0,1]$ is the only possible solution to the first fixed point equation. Recalling from the second one that $\Phi_\infty(0) = \frac{q}{p_0(1-c)}\overline{\Phi}_\infty(1-c)$, we conclude that $\Phi_\infty(0) = 0$ and so $\Phi_\infty(z) = 0$ for all $z \in [0,1]$. The only solution $\Phi_\infty(z)$ is the trivial null one.

It remains to study the cases $c = 1$ and $c = 0$.

• If $c = 1$. In this case, only a single individual can stepwise be removed from the population; the transition matrix P is upper- Hessenberg. This constitutes a simple discrete version of a decay/surge model (some kind of time-reversed version of the simple growth/collapse model).

$$\begin{aligned}\overline{\Phi}_{\infty}(z) &= \frac{p_0 z(z-1)}{z(1-pB(z))-q}\Phi_{\infty}(0)\\ &= \frac{p_0 z(z-1)}{z-1+p(1-zB(z))}\Phi_{\infty}(0).\end{aligned}$$

with $\overline{\Phi}_{\infty}(0) = 0$.

Letting $\overline{B}(z) = (1-B(z))/(1-z)$ the tail generating function of β, this is also

$$\begin{aligned}\overline{\Phi}_{\infty}(z) &= \frac{p_0 z}{1-p\left(1+z\overline{B}(z)\right)}\Phi_{\infty}(0)\\ \Phi_{\infty}(z) &= \left(1+\frac{p_0 z}{1-p\left(1+z\overline{B}(z)\right)}\right)\Phi_{\infty}(0)\end{aligned}$$

With $\overline{B}(1) = B'(1) = \mathbf{E}\beta =: \rho$, $\overline{\Phi}_{\infty}(1) = \frac{p_0}{q-p\rho}\Phi_{\infty}(0) = \frac{1}{1-c}\overline{\Phi}_{\infty}(1-c)$.

We conclude (phase transition):

Subcriticality: $\overline{\Phi}_{\infty}(1) + \Phi_{\infty}(0) = 1 \Rightarrow \Phi_{\infty}(0) = (q-p\rho)/(q-p\rho+p_0)$, well-defined as a probability only if $p\rho < q$. In this case, the chain is positive-recurrent. The term $p\rho$ is the average size of a move-up which has to be smaller than the average size q of a move-down.

Criticality: if $q = p\rho$, $\Phi_{\infty}(0) = 0 \Rightarrow \overline{\Phi}_{\infty}(z) = 0$ for each z. The chain is null-recurrent and it has no non-trivial ($\neq \mathbf{0}$) invariant measure $\boldsymbol{\pi}$.

Supercriticality: $\infty \geq p\rho > q$. The chain is transient at ∞.

Examples:

(i) In addition to $c = 1$, assume $B(z) = \overline{\alpha}z/(1-\alpha z)$ (a geometric model for β with success probability $\overline{\alpha}$). Then, if $p < q\overline{\alpha}$, $\Phi_{\infty}(0) = \pi(0) = (q\overline{\alpha}-p)/(\overline{\alpha}(q+p_0)-p)$ is a probability and

$$\overline{\Phi}_{\infty}(z) = \frac{p_0 z}{1-p(1+z/(1-\alpha z))}\Phi_{\infty}(0) = \frac{p_0 z(1-\alpha z)}{q-z(\alpha+p\overline{\alpha})}\Phi_{\infty}(0).$$

Thus, with $(\alpha < a := (\alpha+p\overline{\alpha})/q < 1)$

$$\pi(x) = [z^x]\,\overline{\Phi}_{\infty}(z) = \pi(0)\frac{p_0}{q}(a-\alpha)\,a^{x-2},\ x \geq 1$$

is the invariant probability measure of the chain, displaying geometric decay at rate a.

(ii) If in addition to $c = 1$, we assume $B(z) = z$, X is reduced to a simple birth and death chain (random walk) on the non-negative integers, reflected at the origin. In this case, we get $\overline{\Phi}_{\infty}(z) = \frac{p_0 z}{1-p(1+z)}\Phi_{\infty}(0)$ with ($\rho = 1$): $\Phi_{\infty}(0) = (q-p)/(q-p+p_0)$. The corresponding chain is positive recurrent if $p < 1/2$, null-recurrent if $p = 1/2$ and transient at ∞ when $p > 1/2$. In the positive-recurrent case, with $\pi(0) = (1-2p)/(1-2p+p_0)$

$$\pi(x) = [z^x]\,\overline{\Phi}_\infty(z) = \pi(0)\,\frac{p_0}{q}\left(\frac{p}{q}\right)^{x-1},\ x \geq 1,$$

a well-known result, [14].

In both examples, whenever the process is positive recurrent, the invariant measure having geometric decay at different rates a and $p/q < 1$ respectively.

• If $c = 0$ (total disasters)

$$\begin{aligned}\overline{\Phi}_\infty(z) &= p_0(z-1)\,\Phi_\infty(0) + pB(z)\,\overline{\Phi}_\infty(z) + q\overline{\Phi}_\infty(1)\\ \Phi_\infty(0) &= \frac{q}{p_0}\overline{\Phi}_\infty(1)\end{aligned}$$

leading to $\Phi_\infty(0) = q/(p_0+q) = \pi(0)$ and

$$\begin{aligned}\overline{\Phi}_\infty(z) &= \frac{p_0 z}{1-pB(z)}\Phi_\infty(0)\\ \Phi_\infty(z) &= \left(1+\frac{p_0 z}{1-pB(z)}\right)\Phi_\infty(0).\end{aligned}$$

The total disaster chain is always positive recurrent with invariant measure related to a shifted compound geometric distribution.

Example:

(i) In addition to $c = 0$, assume $B(z) = \overline{\alpha}z/(1-\alpha z)$ (a geometric model for β with success probability $\overline{\alpha}$). Then

$$\Phi_\infty(z) = \left(1+\frac{p_0 z}{1-pB(z)}\right)\Phi_\infty(0).$$

(ii) In addition to $c = 0$, assume $B(z) = z$. Then, with $\pi(0) = q/(p_0+q)$,

$$\begin{aligned}\Phi_\infty(z) &= \left(1+\frac{p_0 z}{1-pz}\right)\Phi_\infty(0)\\ \pi(x) &= [z^x]\,\Phi_\infty(z) = \pi(0)\,p_0 p^{x-1},\ x \geq 1,\end{aligned}$$

a geometric distribution with decay rate p.

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4. Geometric catastrophes, [3], [4]

The binomial effect is appropriate when the catastrophe affects the individuals in a independent and even way, resulting in a drastic deletion of individuals at each step. The geometric effect would correspond to soft deletion cases where the decline in the population is stopped as soon as any individual survives the catastrophic event. This may be appropriate for some forms of catastrophic epidemics or when the catastrophe has a sequential propagation effect like in the predator-prey models - the predator kills preys until it becomes satisfied (so long as resources are available). More examples can be found in Artalejo et al. [7] and in Cairns and Pollett [8].

The model.

Birth (growth): Let $(\beta_n)_{n\geq 1}$ be an independent identically distributed (iid) sequence taking values in $\mathbb{N} := \{1, 2, ...\}$, with $b_x := \mathbf{P}(\beta_1 = x)$, $x \geq 1$. We shall let $B(z) := \mathbf{E}(z^{\beta})$ be the common probability generating function of the βs.

Death (depletion): Let $(\delta_n)_{n\geq 1}$ be an iid shifted geometric(α) −distributed sequence[1], with success parameter $\alpha \in (0, 1)$. With $\overline{\alpha} := 1 - \alpha$, δ_1 then takes values in $\mathbb{N}_0 := \{0, 1, 2, ...\}$ with $\mathbf{P}(\delta_1 = x) = d_x = \alpha\overline{\alpha}^x$, $x \geq 0$.

Consider the Markov chain with temporal evolution, [4]**:**

$$
\begin{aligned}
X_{n+1} &= X_n + \beta_{n+1} \text{ with probability } p \\
&= (X_n - \delta_{n+1})_+ \text{ with probability } q = 1 - p.
\end{aligned}
\tag{24}
$$

At each step n, the walker moves up with probability p and the amplitude of the upward move is β_{n+1}. The number of step-wise removed individuals (whenever available) is δ_{n+1}, with probability q : the distribution of the sizes of the catastrophes δ_{n+1} does not depend on the current size of the population, so long as there is enough prey. Stated differently, given the population size is x at n, the magnitude of the downward jump is $x \wedge \delta_{n+1}$.

Note that if $X_n = 0$, then $X_{n+1} = \beta_{n+1}$ with probability p (reflection at 0) and $X_{n+1} = 0$ with probability q (absorption at 0).

The one-step stochastic transition matrix P (obeying $P\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is a column vector of ones) of the Markov chain $\{X_n\}$ is:

$$
\begin{aligned}
&P(0,0) = q,\ P(0,y) = pb_y,\ y \geq 1 \\
&P(x,y) = qd_{x-y} = q\alpha\overline{\alpha}^{x-y},\ x \geq 1 \text{ and } 1 \leq y \leq x \\
&P(x,0) = q\textstyle\sum_{k\geq x} d_k = q\overline{\alpha}^x,\ x \geq 1 \text{ and } 0 = y < x \\
&P(x,y) = pb_{y-x},\ x \geq 1 \text{ and } y > x.
\end{aligned}
\tag{25}
$$

With $\boldsymbol{\pi}_n := (\mathbf{P}_{x_0}(X_n = 0), \mathbf{P}_{x_0}(X_n = 1), ...)'$ the column vector[2] of the states' occupation probabilities at time n, $\boldsymbol{\pi}'_{n+1} = \boldsymbol{\pi}'_n P$, $X_0 \overset{d}{\sim} \delta_{x_0}$, is the master equation of its temporal evolution. An equivalent recurrence for the probability generating function of $\{X_n\}$ started at $X_0 = x_0$ is:

---

[1] A geometric(α) rv with success probability α takes values in $\mathbb{N}$. A shifted geometric(α) rv with success probability α takes values in $\mathbb{N}_0$. It is obtained while shifting the former one by one unit.

[2] In the sequel, a boldface variable, say $\mathbf{x}$, will represent a column-vector so that its transpose, say $\mathbf{x}'$, will be a row-vector.

Lemma 1. *With* $\Phi_n(z) := \mathbf{E}_{x_0}\left(z^{X_n}\right) = \sum_{x\geq 0} z^x \mathbf{P}_{x_0}(X_n = x)$, *such that* $\Phi_0(z) = z^{x_0}$, *it holds,*

$$\Phi_{n+1}(z) = \left(pB(z) - \frac{q\alpha z}{\overline{\alpha} - z}\right)\Phi_n(z) + q\left(1 + \frac{\alpha z}{\overline{\alpha} - z}\right)\Phi_n(\overline{\alpha}). \tag{26}$$

Proof: From (25),

$$\begin{aligned}\Phi_{n+1}(z) &= \pi_{n+1}(0) + pB(z)\Phi_n(z) + q\sum_{y\geq 1} z^y \sum_{x\geq y} \pi_n(x)\alpha\overline{\alpha}^{x-y} \\ &= \pi_{n+1}(0) + pB(z)\Phi_n(z) + \frac{q\alpha z}{\overline{\alpha} - z}\left(\Phi_n(\overline{\alpha}) - \Phi_n(z)\right) \\ \Phi_{n+1}(0) &= \pi_{n+1}(0) = q\Phi_n(\overline{\alpha}). \ \square\end{aligned}$$

Remark: The geometric Markov chain is time-homogeneous, irreducible and aperiodic. As a result, all states are either recurrent or transient.

5. The recurrent case

The geometric Markov chain being either transient or recurrent, we shall first exhibit when this transition occurs. We shall then deal first with the recurrent case before switching to the transient regime.

5.1. **Existence and shape of an invariant probability measure.**

Theorem 2. [4]. *The chain* $\{X_n\}$ *with geometric catastrophe is ergodic if and only if* $\rho := \mathbf{E}(\beta_1) < \rho_c := qp^{-1}\frac{\overline{\alpha}}{\alpha} < \infty$. *The probability generating function of* X_∞ *is then given by (27).*

Proof: If a limiting random variable X_∞ exists, from (26), it has probability generating function $\Phi_\infty(z) := \mathbf{E}\left(z^{X_\infty}\right)$ obeying

$$\Phi_\infty(z) = \frac{\overline{\alpha}(1-z)q\Phi_\infty(\overline{\alpha})}{(\overline{\alpha} - z)(1 - pB(z)) + q\alpha z} =: \frac{N(z)}{D(z)}.$$

Suppose $\rho := B'(1) = \mathbf{E}(\beta_1) < \infty$. Only in such a case does the numerator N and denominator D both tend to 0 as $z \to 1$ while the ratio Φ_∞ tends to 1, as required for $\Phi_\infty(z)$ to be a probability generating function. By l'Hospital rule $\frac{N'(z)}{D'(z)} \to 1$ as $z \to 1$, leading to

$$\Phi_\infty(\overline{\alpha}) = \frac{q\overline{\alpha} - p\alpha\rho}{q\overline{\alpha}}.$$

Thus

$$\pi(0) := \mathbf{P}(X_\infty = 0) = \Phi_\infty(0) = q\Phi_\infty(\overline{\alpha}) = q - p\alpha\rho/\overline{\alpha},$$

and, with $\Phi_Y(z) := q/(1 - pB(z))$ the probability generating function of a random variable Y obtained as a shifted-geometric(q) convolution of the βs

$$\begin{aligned}\Phi_\infty(z) &= \frac{(1-z)(q\overline{\alpha} - p\alpha\rho)}{(\overline{\alpha} - z)(1 - pB(z)) + q\alpha z} = \frac{\Phi_Y(z)(1-z)(q\overline{\alpha} - p\alpha\rho)}{(\overline{\alpha} - z)q + q\alpha z\Phi_Y(z)} \\ \pi(x) &: = \pi_\infty(x) = [z^x]\Phi_\infty(z) = \Phi_\infty(0)[z^x]\frac{1-z}{(\overline{\alpha} - z)(1 - pB(z)) + q\alpha z},\ x \geq 1.\end{aligned}$$

If and only if $\rho = \mathbf{E}(\beta_1) < \infty$ and $\zeta := \alpha\left(1 + q^{-1}p\rho\right) < 1$ ($\rho < \rho_c$), then $\pi(0) = \frac{q}{\overline{\alpha}}(1 - \zeta) \in (0,1)$ and the Markov chain is ergodic (positive recurrent and aperiodic). The probability generating function of the invariant probability measure then takes the alternative form

$$\Phi_{\infty}(z) = (1-\zeta)\,\Phi_Y(z)\left[1-\alpha\frac{1-z\Phi_Y(z)}{1-z}\right]^{-1}.$$

Note that, with $\Phi_Z(z) := \frac{1-z\Phi_Y(z)}{(1+m)(1-z)} = \frac{1-\Phi_{Y+1}(z)}{\mathbf{E}(Y+1)(1-z)}$ the probability generating function of some random variable Z, this is also

$$\Phi_{\infty}(z) = \Phi_Y(z)\,\frac{1-\zeta}{1-\zeta\Phi_Z(z)}, \tag{27}$$

which is the product of the probability generating function of Y times the one of a shifted compound geometric$(1-\zeta)$ random variable with compounding probability generating function $\Phi_Z(z)$ (the delay probability generating function of $Y+1$).

Note

$$\zeta < 1 \Leftrightarrow p\mathbf{E}(\beta_1) = p\rho < q\frac{\overline{\alpha}}{\alpha} = q\mathbf{E}(\delta_1),$$

observing $\overline{\alpha}/\alpha = \mathbf{E}(\delta_1)$. This is a sub-criticality condition stating that the one-step unlimited average move down $q\mathbf{E}(\delta_1)$ must exceed the average move up $p\rho$ of the geometric chain. $\square$

Corollary 3. *If $\rho < \rho_c$, the random variable X_{∞} exists and is infinitely divisible (compound Poisson).*

Proof: Consider first the random variable Y with probability generating function $\Phi_Y(z) := \mathbf{E}\left(z^Y\right) = \frac{q}{1-pB(z)}$. It is a compound shifted-geometric random variable with compounding random variable β. It is infinitely divisible because $\Phi_Y(z) = \exp -\lambda(1-\psi(z))$ for some $\lambda > 0$ and some probability generating function $\psi(z)$ with $\psi(0) = 0$. Indeed, with $q =: e^{-\lambda}$, there exists a probability generating function ψ solving $\Phi_Y(z) := q/(1-pB(z)) = \exp -\lambda(1-\psi(z))$. It is

$$\psi(z) = \frac{-\log(1-pB(z))}{-\log q},$$

which is recognized as the probability generating function of a Fisher-log-series random variable. Shifted-geometric convolution random variables are infinitely divisible. For the same reason, the random variable with probability generating function $\frac{1-\zeta}{1-\zeta\Phi_Z(z)}$ is infinitely divisible as a shifted-compound geometric random variable with compounding random variable Z. From (27), $\Phi_{\infty}(z)$ is the product of these two probability generating functions and so is itself the probability generating function of an infinitely divisible random variable X_{∞}. $\square$

Miscellaneous:

- Neuts in [4] observes that if β is itself geometrically distributed (more generally of phase-type), then so are Y, Z, together with X_{∞}, so with an explicit expression of $\pi(x)$. The random variable X_{∞} in particular admits geometric tails. Discrete phase-type random variables are the ones obtained as first hitting times of the absorbing state of a terminating Markov chain with finitely many states, [12].

- In the general case for the law of β, an expression of $\pi(x) := [z^x]\,\Phi_{\infty}(z)$ can be obtained from (27), in principle, using the Faa di Bruno formula ([13], p. 146), observing $\Phi_{\infty}(z) = H(B(z))$ for some generating function H, as a composition of generating functions.

- We also have: $\mu := \mathbf{E}(X_{\infty}) < \infty \Leftrightarrow \mathbf{E}(Y^2) < \infty \Leftrightarrow \mathbf{E}(\beta^2) < \infty$, owing to: $\mathbf{E}(Y^2) = 2\mathbf{E}(Y)^2 + \mathbf{E}(G)\mathbf{E}(\beta^2)$, $\mathbf{E}(G) = p/q$.

Specifically, if $\mathbf{E}(\beta^2) < \infty$

$$\mu = m + \alpha \frac{\mathbf{E}(Y^2) - \mathbf{E}(Y)^2}{2(1-\zeta)} = m\left(1 + \frac{\alpha}{\mathbf{E}(\beta)} \frac{\mathbf{E}(\beta)m + \mathbf{E}(\beta^2)}{2(1-\zeta)}\right).$$

- More generally, it can also be checked that if β only has moments of order $q < 1+a$ $(a > 0)$, then X_{∞} only has moments of order $q < a$.

- Finally, with $m = \Phi_Y'(1) = \mathbf{E}(Y) = q^{-1}p\rho = q^{-1}p\mathbf{E}(\beta)$, as $z \to 1$

$$\frac{1 - z\Phi_Y(z)}{1-z} \sim \frac{1 - z(1 - m(1-z))}{1-z} \sim 1 + m,$$

so that

$$\Phi_{\infty}(1) \sim (1-\zeta)[1 - \alpha(1+m)]^{-1} = 1.$$

- In the positive recurrent case, the geometric process is not time-reversible with respect to the invariant probability measure $\boldsymbol{\pi}$: there is no detailed balance.

- Considerable simplifications are expected when $\beta \overset{d}{\sim} \delta_1$: in this case, the transition matrix P in (25) is of Hessenberg type, that is lower triangular with a non-zero upper diagonal. The corresponding process is skip-free to the right.

5.2. **Time spent in the ground state $\{0\}$.** Whenever the process $\{X_n\}$ is ergodic, it visits infinitely often all the states, in particular the state $\{0\}$, and a sample path of it is made of iid successive non-negative excursions (arches) through that state. By the ergodic theorem, the asymptotic fraction of time spent by $\{X_n\}$ in this state is

$$\frac{1}{N}\sum_{n=1}^{N} \mathbf{1}(X_n = 0) \to \pi(0) = \mathbf{P}(X_{\infty} = 0), \text{ almost surely, as } N \to \infty. \tag{28}$$

Let $\tau_{0,0} := \inf(n \geq 1 : X_n = 0 \mid X_0 = 0)$ be the first return time to state $\{0\}$. By Kac's theorem, [14], its expected value is $\mathbf{E}(\tau_{0,0}) = 1/\pi(0)$ where $\pi(0) = \alpha(1 + q^{-1}p\rho) = \frac{q}{\alpha}(1-\zeta)$.

Suppose $\{X_n\}$ enters state $\{0\}$ from above at some time n_1. The first return time to state $\{0\}$, $\tau_{0,0} := \inf(n > n_1 : X_n = 0 \mid X_{n_1} = 0)$, is:

- either 1 if X_{n_1} stays there with probability $P(0,0) = q$ in the next step; this corresponds to a trivial excursion of length 1 and height 0.

- or, with probability $p = 1-q$, $\{X_n\}$ starts a 'true' excursion with positive height and length $\tau_{0,0}^{+} \geq 2$. Thus,

$$\mathbf{E}(\tau_{0,0}) = \frac{1}{\pi(0)} = 1 + p\mathbf{E}(\tau_{0,0}^{+}) \text{ and}$$

$$\mathbf{E}(\tau_{0,0}^{+}) = \frac{1}{p}\left(\frac{1}{\pi(0)} - 1\right) \geq 2, \tag{29}$$

entailing $\pi(0) < 1/(1+2p)$. Given $\{X_n\}$ enters state $\{0\}$ from above at some time n_1, it stays there with probability $P(0,0) = q$ in the next step, so $\{X_n\}$ will quit state $\{0\}$ at time $n_1 + G$ where G is a geometric random time with success probability $p = 1 - q$. After $n_1 + G$, the chain moves up before returning to state $\{0\}$ again and the time it takes is $\tau_{0,0}^{+}$. Considering two consecutive instants where $\{X_n\}$ enters state $\{0\}$ from above (defining an alternating renewal process), the mean fraction of time θ spent in state $\{0\}$ is:

$$\theta := \frac{\mathbf{E}(G)}{\mathbf{E}(G) + \mathbf{E}\left(\tau_{0,0}^{+}\right)}.$$

From the expression $\mathbf{E}(G) = 1/p$ and the value of $\mathbf{E}\left(\tau_{0,0}^{+}\right)$, consistently with (28), we get:

Proposition 4. *In the positive recurrent case, the mean fraction of time θ spent in state $\{0\}$ is:*

$$\theta = \frac{1}{1 + p\mathbf{E}\left(\tau_{0,0}^{+}\right)} = \pi(0). \tag{30}$$

5.3. **No non trivial ($\neq \mathbf{0}$) invariant measure in the null-recurrent case.** If $\zeta = 1$ (or $\rho = \rho_c$), the critical chain is null-recurrent with

$$\pi(0) = \frac{q\overline{\alpha} - p\alpha\rho}{\overline{\alpha}} = \frac{q}{\overline{\alpha}}(1-\zeta) = 0 \Rightarrow \pi(x) = 0 \text{ for all } x \geq 1.$$

The chain has no non trivial ($\neq \mathbf{0}$) invariant positive measure.

It is not Harris-recurrent, [15], [16] and [17].

5.4. **The generating functional of the geometric model.** With $x_0 \geq 1$, defining the double generating function

$$\Phi_{x_0}(u,z) = \sum_{n\geq 0} u^n \Phi_n(z) = \sum_{n\geq 0} u^n \mathbf{E}_{x_0}\left(z^{X_n}\right),$$

from (26), we get

$$\frac{1}{u}\left(\Phi_{x_0}(u,z) - z^{x_0}\right) = \left(pB(z) - \frac{q\alpha z}{\overline{\alpha} - z}\right)\Phi_{x_0}(u,z) + q\left(1 + \frac{\alpha z}{\overline{\alpha} - z}\right)\Phi_{x_0}(u,\overline{\alpha}),$$

together with

$$\Phi_{x_0}(u,0) = qu\Phi_{x_0}(u,\overline{\alpha}).$$

We obtain

$$\begin{aligned} \Phi_{x_0}(u,z) &= \frac{z^{x_0}(\overline{\alpha} - z) + q\overline{\alpha}u\Phi_{x_0}(u,\overline{\alpha})(1-z)}{(\overline{\alpha} - z)[1 - puB(z)] + q\alpha z u} \\ &= \frac{z^{x_0}(\overline{\alpha} - z) + \overline{\alpha}(1-z)\Phi_{x_0}(u,0)}{(\overline{\alpha} - z)[1 - puB(z)] + q\alpha z u}. \end{aligned} \tag{31}$$

So far, $\Phi_{x_0}(u,z)$ is unknown since it requires the knowledge of $\Phi_{x_0}(u,\overline{\alpha})$ or $\Phi_{x_0}(u,0)$.

Letting $\Phi_{x_0}(u,z) =: N(u,z)/D(u,z)$, the denominator $D(u,z)$ vanishes at

$$u = u(z) = \frac{z - \overline{\alpha}}{q\alpha z + p(z - \overline{\alpha})B(z)}. \tag{32}$$

Note

$$u(1) = 1 \text{ and } u'(1) = \frac{1}{\alpha}(q\overline{\alpha} - \alpha p\rho) = \frac{q}{\alpha}(1-\zeta). \tag{33}$$

The generating function $\Phi_{x_0}(u,z)$ is well-defined when $u < u(z)$ and possibly when $u = u(z)$ as in the recurrent case.

In the recurrent case ($\zeta \leq 1$), $u(z)$ is concave and monotone increasing on the interval $[\overline{\alpha}, 1]$, owing to $u'(1) \geq 0$. The function $u(z)$ has a well-defined inverse $z(u)$ which maps $[0,1]$ to $[\overline{\alpha}, 1]$; this inverse is monotone increasing and convex on this interval. Because in the recurrent case, $[u^n]\Phi_{x_0}(u,z) \to \Phi_\infty(z)$ as $n \to \infty$, $\Phi_{x_0}(u,z)$ also converges as $z \to z(u)$. So both the numerator N and the denominator D of $\Phi_{x_0}(u,z)$ must tend to 0 as $z \to z(u)$, meaning (by L'Hospital rule) that

$$\lim_{z \to z(u)} \frac{N(u,z)}{D(u,z)} = \lim_{z \to z(u)} \frac{N'(u,z)}{D'(u,z)}.$$

Near $z = z(u)$,

$$\begin{aligned}
N(u,z) &= A(z) + B(z)\Phi_{x_0}(u,0) \\
&\sim A(z(u)) + B(z(u))\Phi_{x_0}(u,0) + (z - z(u))\left[A'(z(u)) + B'(z(u))\Phi_{x_0}(u,0)\right] \\
D(u,z) &= C(z) - D(z)u \sim (z - z(u))\left[C'(z(u)) - D'(z(u))u\right],
\end{aligned}$$

imposing $A(z(u)) + B(z(u))\Phi_{x_0}(u,0) = 0$, thereby fixing

$$\Phi_{x_0}(u,0) = \frac{z(u)^{x_0}(z(u) - \overline{\alpha})}{\overline{\alpha}(1 - z(u))} =: G_{x_0,0}(u) \geq 0. \tag{34}$$

Note $G_{x_0,0}(1) = \infty$ and $G_{x_0,0}(0) = \mathbf{P}_{x_0}(X_0 = 0) = 0$. The function

$$G_{x_0,0}(u) = \sum_{n \geq 1} u^n \mathbf{P}_{x_0}(X_n = 0) = \sum_{n \geq 1} u^n P^n(x_0, 0),$$

is the Green kernel of the chain at the endpoints $(x_0, 0)$, [18]. The matrix element $P^n(x_0,0)$ is the contact probability at 0 at time n, starting from x_0.

From (31) and (34), we thus get a closed form expression of $\Phi_{x_0}(u,z)$ when $x_0 \geq 1$, as ($u \in [0,1]$)

Proposition 5. *In the recurrent case ($\zeta \leq 1$), with $z(u)$ the inverse of $u(z)$ defined in (32),*

$$\Phi_{x_0}(u,z) = \frac{z(u)^{x_0}(z(u) - \overline{\alpha})(1-z) - z^{x_0}(z - \overline{\alpha})(1 - z(u))}{(1 - z(u))\left[(\overline{\alpha} - z)(1 - puB(z)) + q\alpha z u\right]}. \tag{35}$$

Remark: If $z = 1$ $\Phi_{x_0}(u,1) = 1/(1-u)$ and $\Phi_{x_0}(0,z) = z^{x_0}$, as required.

5.5. **First return time to 0.** When $x_0 = 0$, observing now $G_{0,0}(0) = \mathbf{P}_0(X_0 = 0) = 1$,

(36)

$$G_{0,0}(u) = 1 + \Phi_0(u,0) = 1 + \frac{z(u) - \overline{\alpha}}{\overline{\alpha}(1 - z(u))} = \frac{\alpha z(u)}{\overline{\alpha}(1 - z(u))}, \text{ with } G_{0,0}(1) = \infty \ (z(1) = 1).$$

This function is the Green kernel at the endpoints $(0,0)$.

If $n\geq 1$, from the recurrence

$$\mathbf{P}_0\left(X_n=0\right)=:P^n\left(0,0\right)=\sum_{m=0}^{n}\mathbf{P}\left(\tau_{0,0}=m\right)P^{n-m}\left(0,0\right),$$

we see, from taking the generating function of both sides and observing the right hand-side is a convolution, that the pgf $\phi_{0,0}\left(u\right)=\mathbf{E}\left(u^{\tau_{0,0}}\right)$ of the first return time to 0, $\tau_{0,0}$ and $G_{0,0}\left(u\right)$ are related by the Feller relation (see [19] and [20] pp $3-4$ for example)**:** $G_{0,0}\left(u\right)=1+\ G_{0,0}\left(u\right)\phi_{0,0}\left(u\right)$**.** Hence, with $\phi_{0,0}\left(0\right)=0$,

$$\phi_{0,0}\left(u\right)=\mathbf{E}\left(u^{\tau_{0,0}}\right)=1-\frac{1}{G_{0,0}\left(u\right)}=\frac{z\left(u\right)-\overline{\alpha}}{\alpha z\left(u\right)},\ \phi_{0,0}\left(1\right)=1. \tag{37}$$

In particular, observing $z'\left(1\right)=1/u'\left(1\right)=\frac{\alpha}{q(1-\zeta)}$, we get

$$\begin{aligned}\mathbf{E}\left(\tau_{0,0}\right) &= \overline{\alpha}z'\left(1\right)/\alpha=\frac{\overline{\alpha}}{q\left(1-\zeta\right)}\text{ if }\zeta<1\text{ (positive recurrence)},\\ &= \infty\text{ if }\zeta=1\text{ (null recurrence)}.\end{aligned}$$

Note as required from Kac's theorem: $\mathbf{E}\left(\tau_{0,0}\right)=1/\pi\left(0\right).$

5.6. **Contact probability at** 0 **and first time to extinction.** Also, with $x_0\geq 1$ ($\phi_{x_0,0}\left(0\right)=0$), using (34) and (36),

$$\mathbf{E}\left(u^{\tau_{x_0,0}}\right)=\phi_{x_0,0}\left(u\right)=\frac{G_{x_0,0}\left(u\right)}{G_{0,0}\left(u\right)}=\frac{1}{\alpha}z\left(u\right)^{x_0-1}\left(z\left(u\right)-\overline{\alpha}\right)=\phi_{0,0}\left(u\right)z\left(u\right)^{x_0}, \tag{38}$$

gives the probability generating function of the first hitting time of 0, starting from $x_0\geq 1$ (the first extinction time of the chain). We also get

$$\begin{aligned}\mathbf{E}\left(\tau_{x_0,0}\right) &= \frac{z'\left(1\right)}{\alpha}\left(\alpha x_0+\overline{\alpha}\right)=\frac{\alpha x_0+\overline{\alpha}}{q\left(1-\zeta\right)}\text{ if }\zeta<1\text{ (positive recurrence)},\\ &= \infty\text{ if }\zeta=1\text{ (null recurrence)}.\end{aligned}$$

If $z^{''}\left(1\right)<\infty$ ($\sigma^2\left(\beta\right)<\infty$), the variance of $\tau_{x_0,0}$ in the positive-recurrent case is finite and found to be

$$\begin{aligned}\sigma^2\left(\tau_{x_0,0}\right) &= \frac{z^{''}\left(1\right)+z'\left(1\right)}{\alpha}\left(\alpha x_0+\overline{\alpha}\right)-\frac{z'\left(1\right)^2}{\alpha^2}\left(\alpha^2x_0+1-\alpha^2\right)\\ &= x_0\left(z^{''}\left(1\right)+z'\left(1\right)-z'\left(1\right)^2\right)+\frac{\overline{\alpha}}{\alpha^2}\left(\alpha\left(z^{''}\left(1\right)+z'\left(1\right)\right)-\left(1+\alpha\right)z'\left(1\right)^2\right).\end{aligned}$$

If $\sigma^2\left(\beta\right)=\infty$, $\sigma^2\left(\tau_{x_0,0}\right)=\infty$.

We observe from (37)

$$\phi_{0,0}\left(u\right)=\mathbf{E}\left(u^{\tau_{0,0}}\right)=\frac{z\left(u\right)-\overline{\alpha}}{\alpha z\left(u\right)},\text{ entailing }z\left(u\right)=\frac{\overline{\alpha}}{1-\alpha\phi_{0,0}\left(u\right)}.$$

The function $\phi_{0,0}\left(u\right)$ being a probability generating function, $z\left(u\right)$ is the probability generating function of some random variable $\tau\geq 0$; namely, the one of a compound shifted-geometric($\overline{\alpha}$) probability generating function with compounding probability generating function $\phi_{0,0}(u)$. And $z(u)^{x_0}$ is the probability generating function of the random variable

$$\tau_{x_0} = \sum_{k=1}^{x_0} \tau_k, \tag{39}$$

where τ_k is an iid sequence with $\tau_k \sim \tau$. We have thus proven

Theorem 6. *In the recurrent case, the function $z(u)$ is the probability generating function of some random variable $\tau \geq 0$. With τ_{x_0} defined in (39), the first time to extinction random variable $\tau_{x_0,0}$ is decomposable as*

$$\tau_{x_0,0} \overset{d}{=} \tau_{x_0} + \tau_{0,0}, \tag{40}$$

where $\tau_{x_0} \geq 0$ and $\tau_{0,0} \geq 1$ are independent. With $\phi_{0,0}(u)$ given by (37), the probability generating function of $\tau_{x_0,0}$ is given in (38) with $z(u) = \overline{\alpha}/\left(1 - \alpha\phi_{0,0}(u)\right)$.

The function $u(z)$ is explicitly defined in (32) and $z(u)$ can be obtained from the Lagrange inversion formula as follows: develop $u(z)$ as a power series in $\widehat{z} := (z - \overline{\alpha})/\alpha$, where

$$u(\widehat{z}) = \frac{\widehat{z}}{q\overline{\alpha} + \widehat{z}\left(q\alpha + pB\left(\overline{\alpha} + \alpha\widehat{z}\right)\right)}$$

maps $[0,1]$ to $[0,1]$. It can in principle be obtained by Faa di Bruno formula for the composition of analytic functions ([13], p. 146), observing $\widehat{z}^{-1}u(\widehat{z}) = u_1(u_2(\widehat{z}))$ where $u_1(z) = 1/(q\overline{\alpha} + z)$ and $u_2(\widehat{z}) = \widehat{z}\left(q\alpha + pB\left(\overline{\alpha} + \alpha\widehat{z}\right)\right)$ both with well-defined series expansion near 0. Then, by using Lagrange inversion formula ([13], p. 159) we can get $\widehat{z}(u)$ satisfying $u(\widehat{z}(u)) = u$ and then also the output

$$\mathbf{E}\left(u^{\tau_{x_0,0}}\right) = \frac{1}{\alpha} z(u)^{x_0-1}\left(z(u) - \overline{\alpha}\right) = \widehat{z}(u)\left(\alpha\widehat{z}(u) + \overline{\alpha}\right)^{x_0-1}.$$

The probability generating function $z(u)$ is well-defined in the range $u \in [0,1]$ in the recurrent case and $u \in [0,1] \to z(u) \in [\overline{\alpha}, 1]$ is its range.

5.7. **Harmonic (scale) function and the height of an excursion (recurrent case).** Suppose $\{X_n\}$, with transition matrix P, is recurrent. Let then $P \to \overline{P}$ with $\overline{P}(0,y) = \delta_{0,y}$, forcing state 0 to be absorbing, corresponding to the substitution of processes: $X_n \to Y_n := X_{n\wedge\tau_{x,0}}$ (the process X_n started at x_0 and stopped when it first hits 0). The matrix $\overline{P}$ is stochastic but non irreducible, having $\{0\}$ as an absorbing class. For the process Y_n with transition matrix $\overline{P}$, eventually $Y_\infty = 0$ with probability 1. The harmonic (or scale) sequence $\boldsymbol{\phi}$ solves:

$$\overline{P}\boldsymbol{\phi} = \boldsymbol{\phi}, \tag{41}$$

where $\boldsymbol{\phi} = (\phi(0), \phi(1), ...)'$ is the right column eigenvector of $\overline{P}$ associated to the eigenvalue 1. With $h \gg 1$ and $x \in \{1, ..., h-1\}$, let

$$\begin{aligned} \tau_{x,h} &= \inf\left(n \geq 0 : X_n \geq h \mid X_0 = x\right) \text{ if such an } n \text{ exists,} \\ &= +\infty \text{ if not.} \end{aligned}$$

By induction, with $\phi(0) := 0$ and $\tau_x^{(h)} := \tau_{x,0} \wedge \tau_{x,h}$

$$\forall x \in \{1, ..., h-1\}, \forall n \geq 0\text{: } \phi(x) = \mathbf{E}\phi\left(Y_{n\wedge\tau_{x,h}}\right) = \mathbf{E}\phi\left(X_{n\wedge\tau_x^{(h)}}\right).$$

The harmonic function ϕ on $\{1,...,h\}$, makes $\phi\left(X_{n\wedge\tau_x^{(h)}}\right)$ a martingale. Therefore, as $n\to\infty$, $\forall x\in\{1,...,h-1\}$:

$$\phi(x)=\mathbf{E}\phi\left(X_{\tau_x^{(h)}}\right)=\phi(0)\,\mathbf{P}(\tau_{x,0}<\tau_{x,h})+\mathbf{E}\phi\left(X_{\tau_{x,h}}\right)\mathbf{P}(\tau_{x,h}<\tau_{x,0}).\tag{42}$$

As $h\to\infty$, both $\tau_{x,h}$ and the overshoot $X_{\tau_{x,h}}\to\infty$. Assuming there is a solution $\phi(x)\geq 0$ such that $\phi(0)=0$ and $\phi(x)\to\infty$ as $x\to\infty$, owing to $X_{\tau_{x,h}}\to\infty$, yields

$$\mathbf{P}(\tau_{x,h}<\tau_{x,0})=\frac{\phi(x)}{\mathbf{E}\phi\left(X_{\tau_{x,h}}\right)}\underset{h\to\infty}{\to}0=\mathbf{P}(\tau_{x,\infty}<\tau_{x,0}),\tag{43}$$

indeed consistent with the guess: Y_n does not escape, it goes extinct with probability 1, whatever the initial condition x is.

Let us now consider the problem of evaluating the law of the height H of an excursion in the recurrent case. Firstly, $H=0$ with probability q (corresponding to a trivial excursion). Now, for non-trivial excursions, the event $H\geq h\geq 1$ is realized whenever a first birth event occurs with size $\beta_1\geq h$ or, if $0<\beta_1<h$, when the time (starting from β_1) needed to first hit $\{h,h+1,...\}$ is less than the time needed to first hit 0, namely the event $\tau_{\beta_1,h}<\tau_{\beta_1,0}$. Hence,

$$\mathbf{P}(H\geq h)=p\mathbf{P}(\beta_1\geq h)+p\sum_{x=1}^{h-1}\mathbf{P}_0(\beta_1=x)\,\mathbf{P}(\tau_{x,h}<\tau_{x,0}),$$

where $\mathbf{P}(\beta_1=x)=b_x$, $\mathbf{P}(\beta_1\geq h)=\sum_{y\geq h}b_y$ and $\mathbf{P}(\tau_{x,h}<\tau_{x,0})$ is given in (43). Note $\mathbf{P}(H\geq 1)=p$ and, as required since $\mathbf{P}(\tau_{x,\infty}<\tau_{x,0})=0$, $H<\infty$ with probability 1.

6. The transient case

In the transient case ($\zeta>1$ or $\rho>\rho_c$), the denominator of $\Phi_{x_0}(u(z),z)$ tends to 0 as $u\to u(z)$ and $u(z)$ does not cancel the numerator: $u(z)$ is a true pole of $\Phi_{x_0}(u,z)$. When $z\in[\overline{\alpha},1]$, $u(z)$ is concave but it has a maximum $u(z_*)$ strictly larger than 1, attained at some z_* inside $(\overline{\alpha},1)$, owing to $u(\overline{\alpha})=0$, $u(1)=1$ and $u'(1)<0$. We anticipate that for some constant $C>0$ depending on x_0,

$$\mathbf{P}_{x_0}(X_n=0)\sim C\cdot u(z_*)^{-n}\ \text{ as } n\to\infty,$$

stating that $\{X_n\}$ only visits 0 a finite number of times before drifting to ∞.

6.1. **No non trivial ($\neq \mathbf{0}$) invariant measure in the transient case.** Before turning to this question, let us observe the following: suppose $\zeta>1$ ($\rho>\rho_c$), so that the super-critical geometric chain is transient. For the same reason as for the null-recurrent case, the chain has no non trivial ($\neq \mathbf{0}$) invariant positive measure either. It is not Harris-transient, [16].

6.2. **Large deviations.** Consider $v(z) := 1/u(z)$ where $u(z)$, as a pole, cancels the denominator of $\Phi_{x_0}(u, z)$ without cancelling its numerator, so

$$v(z) = \frac{q\alpha z + p(z - \overline{\alpha}) B(z)}{z - \overline{\alpha}}.$$

Over the domain $1 \geq z > \overline{\alpha}$, $v(z)$ is convex with

$$v'(z) = pB'(z) - q\alpha\overline{\alpha}(z - \overline{\alpha})^{-2} \text{ and } v''(z) = pB''(z) + 2q\alpha\overline{\alpha}(z - \overline{\alpha})^{-3} > 0.$$

Thus

$$v(1) = 1 \text{ and } v'(1) = \frac{p\alpha\rho - q\overline{\alpha}}{\alpha} = p\rho - q\overline{\alpha}/\alpha.$$

We have $v'(1) > 0$ if and only if the chain is transient. In this transient case,

$$\Phi_n(z)^{1/n} \to v(z) \text{ as } n \to \infty.$$

Define

$$F(\lambda) := -\log v\left(e^{-\lambda}\right) = \log u\left(e^{-\lambda}\right).$$

The function $F(\lambda)$ is concave on its definition domain $\lambda \in [0, -\log\overline{\alpha})$. It is first increasing, attains a maximum and then decreases to $-\infty$ while crossing zero in between. There exists $\lambda^* \in (0, -\log\overline{\alpha})$ such that $\omega_* = F'(\lambda^*) = 0$.

With $\omega \in (F'(-\log\overline{\alpha}), F'(0)]$, define

$$f(\omega) = \inf_{0 \leq \lambda < -\log\overline{\alpha}} (\omega\lambda - F(\lambda)) \leq 0, \tag{44}$$

the Legendre conjugate of $F(\lambda)$. The variable ω is Legendre conjugate to λ with $\omega = F'(\lambda)$ and $\lambda = f'(\omega)$. Note $F'(-\log\overline{\alpha}) = -\infty$ and $F'(0) = v'(1) > 0$. On its definition domain, $f(\omega) \leq 0$ is increasing and concave, starting from $f(-\infty) = -\infty$ and ending with $f(F'(0)) = 0$ where $f'(F'(0)) = 0$.

From [21], we get:

Proposition 7. *For those ω in the range $[\omega_* = 0, F'(0)]$ and for any $x_0 > 0$,*

$$\frac{1}{n}\log \mathbf{P}_{x_0}\left(\frac{1}{n}X_n \leq \omega\right) \underset{n\to\infty}{\to} f(\omega). \tag{45}$$

In particular, at $\omega = F'(0) = v'(1) > 0$ where $f(F'(0)) = -F(0) = 0$, we get

$$\frac{1}{n}X_n \overset{a.s.}{\to} v'(1) \text{ as } n \to \infty.$$

To keep $\omega = F'(\lambda)$ in the non-negative range $[\omega_* = 0, F'(0)]$, the range of λ should then equivalently be restricted to $[0, \lambda^*]$. We clearly have $f(\omega_*) = -F(\lambda^*) < 0$ and, from (45),

$$-\frac{1}{n}\log \mathbf{P}_{x_0}\left(\frac{1}{n}X_n \leq 0\right) = -\frac{1}{n}\log \mathbf{P}_{x_0}(X_n \leq 0) \to F(\lambda^*) \text{ as } n \to \infty. \tag{46}$$

This shows the rate at which $\mathbf{P}_{x_0}(X_n = 0)$ decays exponentially with n. Equivalently, with $z_* = e^{-\lambda^*}$, $\mathbf{P}(X_n = 0) \sim C \cdot u(z_*)^{-n}$ as guessed and the Green series $\Phi_{x_0}(1, 0) = G_{x_0,0}(1) = \sum_{n\geq 1}\mathbf{P}_{x_0}(X_n = 0)$ now is summable for all $x_0 \geq 0$ (translating that $\{X_n\}$ visits 0 only a finite number of times). In the transient case,

$$\mathbf{P}(\tau_{x_0,0} < \infty) = \phi_{x_0,0}(1) = \frac{G_{x_0,0}(1)}{G_{0,0}(1)} < 1.$$

6.3. **The scale function and the extinction probability.** In the transient case, the chain $\{X_n\}$ started at $x>0$ can drift to ∞ before it first hits 0. There is thus only a probability smaller than 1 that $\{X_n\}$ gets extinct for the first time.

In the transient case, let then $P \to \overline{P}$ with $\overline{P}(0,y)=\delta_{0,y}$ forcing state 0 to be absorbing, corresponding to: $X_n \to Y_n := X_{n\wedge \tau_{x,0}}$. The matrix $\overline{P}$ is stochastic but non irreducible, having an absorbing class. The harmonic (or scale) sequence $\boldsymbol{\phi}$ solves:

$$\overline{P}\boldsymbol{\phi}=\boldsymbol{\phi}, \tag{47}$$

where $\boldsymbol{\phi}=(\phi(0),\phi(1),...)'$ is a column vector. With $h \gg 1$ and $x \in \{1,...,h-1\}$, let

$$\begin{aligned} \tau_{x,h} &= \inf(n\geq 0 : X_n \geq h \mid X_0 = x) \text{ if such an } n \text{ exists,} \\ &= +\infty \text{ if not.} \end{aligned}$$

By induction, with $\phi(0)=1$ and $\tau_x^{(h)} := \tau_{x,0}\wedge\tau_{x,h}$

$$\forall x\in\{1,...,h-1\},\ \forall n\geq 0\text{: } \phi(x)=\mathbf{E}\phi\left(Y_{n\wedge\tau_{x,h}}\right)=\mathbf{E}\phi\left(X_{n\wedge\tau_x^{(h)}}\right).$$

The harmonic function on $\{1,...,h\}$, makes $\phi\left(X_{n\wedge\tau_x^{(h)}}\right)$ a martingale, [22]. As $n\to\infty$, $\forall x\in\{1,...,h-1\}$:

$$\phi(x)=\mathbf{E}\phi\left(X_{\tau_x}\right)=\phi(0)\,\mathbf{P}\left(\tau_{x,0}<\tau_{x,h}\right)+\mathbf{E}\phi\left(X_{\tau_{x,h}}\right)\mathbf{P}\left(\tau_{x,h}<\tau_{x,0}\right). \tag{48}$$

As $h\to\infty$, both $\tau_{x,h}$ and the overshoot $X_{\tau_{x,h}}\to\infty$. Assuming there is a solution $\phi(x)>0$ such that $\phi(x)\to 0$ as $x\to\infty$, yields

$$\mathbf{P}\left(\tau_{x,0}<\tau_{x,\infty}\right)=\phi(x), \tag{49}$$

indeed consistent with the guess: $\phi>0$ and vanishing at ∞. This expression also shows that $\phi(x)$ must be decreasing with x. Note $\mathbf{P}\left(\tau_{x,0}<\tau_{x,\infty}\right)=\mathbf{P}\left(X_{\tau_x}=0\right)$ where $\tau_x := \tau_{x,0}\wedge\tau_{x,\infty}$. We obtained:

Proposition 8. *In the transient case, with $\boldsymbol{\phi}$ defined in (47) obeying $\phi(0)=1$,*

$$\phi(x)=\mathbf{P}\left(\tau_{x,0}<\tau_{x,\infty}\right),$$

is the probability of extinction starting from state x.

Remark: The function $\overline{\boldsymbol{\phi}} := \mathbf{1}-\boldsymbol{\phi}$ clearly also is a (increasing) harmonic function so that so is any convex combination of $\boldsymbol{\phi}$ and $\overline{\boldsymbol{\phi}}$, [20]. Clearly, $\overline{\phi}(x) = \mathbf{P}\left(\tau_{x,\infty}<\tau_{x,0}\right)=\mathbf{P}\left(X_{\tau_x}=\infty\right)$.

6.4. **The height of an excursion (transient case).** The law of the height H of an excursion is given by

$$\begin{aligned} \mathbf{P}(H\geq h) &= \mathbf{P}_0\left(X_1\geq h\right)+\sum_{x=1}^{h-1}\mathbf{P}_0\left(X_1=x\right)\mathbf{P}\left(X_{\tau_x}\geq h\right) \\ &= \mathbf{P}_0\left(X_1\geq h\right)+\sum_{x=1}^{h-1}\mathbf{P}_0\left(X_1=x\right)\mathbf{P}\left(\tau_{x,h}<\tau_{x,0}\right), \end{aligned}$$

where $\mathbf{P}_0(X_1 = x) = P(0,x) = pb_x$, $\mathbf{P}_0(X_1 \geq h) = p\sum_{y\geq h} b_y$ and, from (48),

$$\mathbf{P}(\tau_{x,h} < \tau_{x,0}) = \frac{1-\phi(x)}{1-\mathbf{E}\phi\left(X_{\tau_{x,h}}\right)}.$$

Note, as required, that $H = 0$ with probability q and

$$H = \infty \text{ with probability } \sum_{x\geq 1} \mathbf{P}_0(X_1 = x)\,\overline{\phi}(x).$$

6.5. **Doob transform: conditioning on non-extinction.** In the transient case when the chain can either drift to infinity or go extinct, the harmonic sequence $\boldsymbol{\phi}$ plays some additional role in a conditioning. With $D_{\boldsymbol{\phi}}$:=diag$(\phi(0), \phi(1), ...)$, the matrix

$$P_{\boldsymbol{\phi}} := D_{\boldsymbol{\phi}}^{-1}\overline{P}D_{\boldsymbol{\phi}}$$

is a stochastic matrix. We have $P_{\boldsymbol{\phi}}(m,0) = \boldsymbol{\delta}_{m,0}$ so that state 0 is inaccessible from any other state than 0 itself: state 0 is isolated and disconnected and so may be removed from the state space. We have ([23], page 327, [22]) :

Proposition 9. *In the transient case, $P_{\boldsymbol{\phi}}$ is the one-step transition matrix corresponding to the process $X_{n\wedge\tau_x}$ conditioned not to be absorbed at* 0.

Remark: this is a selection of paths procedure allowing to focus only on those paths of the transient chain that do not go extinct.

7. Concluding remarks

We revisited the geometric catastrophe model in discrete-time, as a Markovian population dynamics on the non-negative integers. For this process, a collapse move from some state is geometrically distributed so long as it does not exhaust the current value of this state. It is balanced by random growth moves with arbitrary distribution. This process may be viewed as a generalized birth and death chain. Using generating function techniques, conditions under which the birth and death competing events yield a process which is stable (recurrent) have been highlighted. When recurrent, the shape of the invariant probability measure was described. When unstable (transient), the chain either drifts to infinity or go extinct, a feature similar to supercritical branching processes, [24]. The height and length of excursions, extinction probability, time to extinction, have been studied both in the recurrent and transient setups. The harmonic (scale) function was shown to play an important role in the analysis.

Other interesting random walks in the same spirit, but with different collapse rules, were introduced in [4]. We plan, as we do here for the truncated geometric case, to lift the veil on some of their intrinsic statistical properties which are expected to be of a completely different nature.

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T. Huillet: Laboratoire de Physique Théorique et Modélisation, CY Cergy Paris Université, CNRS UMR-8089, 2 avenue Adolphe-Chauvin, 95302 Cergy-Pontoise, FRANCEThierry.Huillet@cyu.fr